Burning Ropes to Measure Time*

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Each rope burns from one end to the other in exactly one hour, but at an uneven rate. Which fractions of an hour are measureable by burning several ropes from one or both ends?

The Associate Editor of my previous paper [1] drew my attention to another popular puzzle discussed on many websites. With due thanks, I quote the puzzle from InterviewBit [2]. Then I solve it, generalize it, and dig deeper to reveal some mathematical truths.

1. The Rope Burning Puzzle

You have two ropes and a lighter. See *Figure* 1. Each rope takes exactly one hour to burn, but not necessarily evenly; that is, the first half might burn in the first 10 minutes and the second half in 50 minutes. Mark all times you can measure by burning these two ropes.

A 1 hour B 30 minutes C 45 minutes D 15 minutes

1.1 *R*eaders' Role

To derive optimal benefit from this paper, as in [1], I suggest readers proceed as follows:

(1) Read the paper from beginning to end;

(2) Pause when I raise a question;

(3) Answer the question—first individually, and then in collaboration with one or more partners; and

Keywords
Puzzle, bisectional functions, mathematical induction, proving impossibility, limits, approximations, fixed point, continued fraction.

*DOI: https://doi.org/10.1007/s12045-019-0910-5

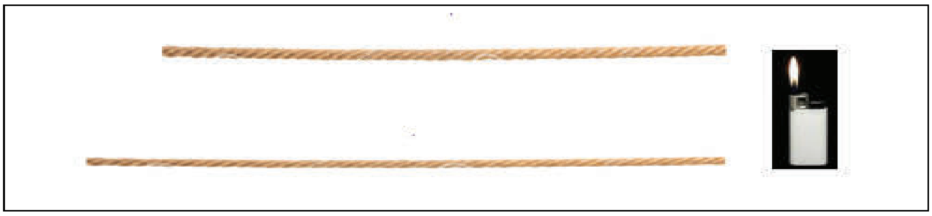

Figure 1. Two ropes burn for an hour each, but at uneven rates.

(4) Read my solution and propose alternative solutions. You may skip the proofs, at least on first reading.

1.2 *Solution*

Here is the solution to the interview question: All four alternatives are measurable times! In fact, one rope suffices to measure one hour while it burns from end to end, and half an hour while it burns simultaneously from both ends. If two ropes are available, ignite both ends of Rope 1 and one end of Rope 2. When Rope 1 burns out, half an hour has passed, and half an hour remains on Rope 2. At this epoch, ignite the other end of Rope 2. When Rope 2 burns out, 3/4 hour has passed starting from the beginning, and 1/4 hour starting from the half hour epoch.

1.3 *Generalized Rope Burning Puzzle*

Let us extend the puzzle to more than two ropes. What are the proper fractions of an hour that you can measure using these ropes? Clearly, you can measure any improper fraction of the form $m + n/d$ by sequentially burning m more ropes right after measuring the proper fraction n/d of an hour. Therefore, in this paper, we deal with proper fractions only.

You can measure any improper fraction, if you can measure the proper part of it.

In Section 2, we allow a finite number of ropes, each of which burn in exactly one hour, but at unknown, uneven and unequal rates. In Section 3, we allow an unlimited supply of such ropes.

2. Finite Number of Ropes

Suppose that you are at a picnic spot where electricity is available. You want to cook khichdi (a porridge of rice, lentils and veggies) in a microwave. You put all ingredients in a pot; you put the pot in the microwave; you are ready to set the timer to 9/16 of an hour only to realize that the timer on the microwave is not working! You do not have a watch or any other timer; but you have a lighter and one dozen ropes, which will burn from one end to the other in exactly one hour each, though at uneven rates. How can you burn the ropes to determine when to press the 'Start' button and when to press the 'End' button on the microwave to cook your food in 9/16 of an hour? You must abide by the following rules:

Rule 1 You may ignite one end of a rope, and immediately or sometime later you may ignite its other end;

Rule 2 Once ignited, you may not extinguish a flame;

Rule 3 You may not ignite a rope from one or more interior points, since you cannot ensure when any one piece of the rope will completely burn out.

You may not ignite a rope from one or more interior points.

Here is a solution to measure exactly 9/16 of an hour: Ignite Rope 1 on both ends, Ropes 2 and 3 each on one end. When Rope 1 burns out (1/2 hour gone), ignite the other end of Rope 2, let Rope 3 continue to burn from one end, and ignite Rope 4 on one end. When Rope 2 burns out (another 1/4 hour gone, and Rope 3 has been burning for 3/4 hour), ignite the other end of Rope 3, let Rope 4 continue to burn from one end, and ignite Rope 5 on one end. When Rope 3 burns out (another 1/8 hour gone, and Rope 4 has been burning for 3/8 hour), ignite the other end of Rope 4, and let Rope 5 continue to burn as before. When Rope 4 burns out (yet another 5/16 of an hour is gone, and Rope 5 has been burning for 7/16 hour), start measuring time; and end measuring time when Rope 5 burns out completely. The measured time is 9/16 of an hour.

Although it is a correct solution, is it an efficient solution? Can we not solve the problem with fewer ropes?

Although it is a correct solution, is it an efficient solution? Can we not solve the problem with fewer ropes?

In order to write down any proposed solution in a compact manner, let us develop a shorthand notation.

2.1 *A Shorthand Notation for the Solution*

Suppose that only two ropes are available. To measure 3/4 hour and 1/4 hour, we write down a solution in three steps. First, we write down which ropes we burn, and when, as

$$|\ \underline{\overline{1}}\ \overline{2}\ |\ \underline{\overline{2}}\ |$$

An overbar means a rope is burning from one end; both over and under bars mean a rope is burning from both ends.

This notation begins with a vertical line and shows between successive vertical lines the labels of ropes that we burn either from one end (shown by an overbar), or from both ends (shown by both over and under bars). Each new vertical line indicates that we have waited until one of the ropes has burnt out completely; and we are ready to ignite the other end of one or more already burning ropes, or one or both ends of one or more new ropes.

Second, we evaluate the elapsed times between successive vertical lines. In the above example, these times are (1/2, 1/4). At each vertical line, we also calculate how much time remains, if any, on each rope to aid the upcoming calculations.

The end epoch is when all ropes burn out fully.

Third, we declare the start epoch and the end epoch to measure the desired time duration. Clearly, the end epoch is when all ropes burn out fully (for otherwise, we can eliminate those ropes that are still burning when we stop measuring time). So, we replace the last vertical line by a closing brace }. To denote the start time, we must convert one of the previous vertical lines to an opening brace {. For example, we measure 3/4 hour and 1/4 hour respectively as

$$\{\ \underline{\overline{1}}\ \overline{2}\ |\ \underline{\overline{2}}\ \} \text{ and } |\ \underline{\overline{1}}\ \overline{2}\ \{\ \underline{\overline{2}}\ \}$$

Henceforth, we shall refer to a vertical line, an opening brace and a closing brace by the common word 'separator.' Using this shorthand notation, we write the solution to the microwave problem of measuring 9/16 of an hour as

$$S_1 = \mid \underline{\bar{1}}\ \bar{2}\ \bar{3} \mid \underline{\bar{2}}\ \bar{3}\ \bar{4} \mid \underline{\bar{3}}\ \bar{4}\ \bar{5} \mid \underline{\bar{4}}\ \bar{5}\ \{\ \bar{5}\ \}$$

The times between separators are $(1/2, 1/4, 1/8, 5/16, 9/16)$. Hence, the measured time between braces is 9/16 of an hour.

The opening brace { denotes the start time, and the closing brace } denotes the end time.

Alternatively, for $S_2 = \mid \underline{\bar{1}}\ \bar{2}\ \bar{3}\ \bar{4} \mid \underline{\bar{2}}\ \bar{3}\ \bar{4} \mid \underline{\bar{3}}\ \bar{4}\ \{\ \underline{\bar{4}} \mid \underline{\bar{5}}\ \}$, the times between separators are $(1/2, 1/4, 1/8, 1/16, 1/2)$. Hence, the measured time between braces is $1/16 + 1/2 = 9/16$ of an hour. Note that while in the first solution S_1, Rope 5 burns from one end only; in the second solution S_2, it burns from both ends. Nonetheless, both solutions require five ropes. Can we find a solution that requires fewer ropes?

Can we find a solution that requires fewer ropes?

Yes, we can. A third solution $S_3 = \mid \underline{\bar{1}}\ \bar{2}\ \bar{3} \mid \underline{\bar{2}}\ \bar{3}\ \{\ \underline{\bar{3}}\ \bar{4} \mid \underline{\bar{4}}\ \}$, with times between successive separators$(1/2, 1/4, 1/8, 7/16)$, measures $1/8 + 7/16 = 9/16$ of an hour between braces. This solution, using only four ropes, is more efficient than the previous two solutions!

Encouraged, we become more ambitious and ask, "Is S_3 the most efficient solution; or, is there a solution with only three ropes?"

Along with this question, call it Q1, let us pose several other questions that will motivate the rest of this paper. You should answer Questions Q2–Q4 using **as few ropes** as you absolutely need.

Q1 How can you measure 9/16 of an hour using three ropes?

Q2 How can you measure 40 minutes (or 2/3 hour)?

Q3 How can you measure approximately 40 minutes within an error of five seconds?

Q4 How can you measure approximately 35 minutes (or 7/12 of an hour) within an error of one second?

Answer these questions using as few ropes as necessary.

Set aside this paper; and answer the above questions on your own, or jointly with one or more partners. When you return to the paper, you will find answers to Questions Q1 and Q2 in Subsection 2.6, Q3 in Subsection 2.7, and Q4 in Subsection 3.2. First, we must develop some necessary concepts.

Set aside the paper; and answer the questions.

2.2 *Proper Fractions that are n-rope-Measurable*

Let us list which proper fractional hours are measurable with one, two and three ropes.

1) With one rope, we can measure 1 hour and 1/2 hour as $\{\ \bar{1}\ \}$ and $\{\ \underline{\bar{1}}\ \}$ respectively.

2) With two ropes, we can also measure 1/4 and 3/4 hour as $|\ \underline{\bar{1}}\ \bar{2}\ \{\ \bar{2}\ \}$ and $\{\ \underline{\bar{1}}\ \bar{2}\ |\ \underline{\bar{2}}\ \}$ respectively.

3) With three ropes, we can further measure 1/8, 3/8, 5/8 and 7/8 hour respectively as $|\ \underline{\bar{1}}\ \bar{2}\ \bar{3}\ |\ \underline{\bar{2}}\ \bar{3}\ \{\ \underline{\bar{3}}\ \}$; $|\ \underline{\bar{1}}\ \bar{2}\ |\underline{\bar{2}}\ \bar{3}\ \{\ \underline{\bar{3}}\ \}$; $|\ \underline{\bar{1}}\ \bar{2}\ \{\ \underline{\bar{2}}\ \bar{3}\ |\ \underline{\bar{3}}\ \}$; and $\{\ \underline{\bar{1}}\ \bar{2}\ \bar{3}\ |\ \underline{\bar{2}}\ \bar{3}\ |\ \underline{\bar{3}}\ \}$.

Which proper fractions are n-rope measurable?

What are the proper fractions of an hour measurable with $n \geq 4$ ropes? Proposition 2.2.1, which relies on Lemma 2.2.1, answers this question. We call such proper fractions n-rope-measurable.

Lemma 2.2.1 If x is a proper fraction of an hour measurable with n ropes, then when $(n + 1)$ ropes are given, we can also measure proper fractions $(1 - x)/2$ and $(1 + x)/2$.

Proof. At the start epoch S of x, ignite one end of Rope $(n + 1)$. At the end epoch E of x, when the time remaining on Rope $(n+1)$ is $(1-x)$, ignite also the other end of Rope $(n+1)$. If Rope $(n+1)$ burns out completely at epoch F, then the duration EF is $(1-x)/2$, and the duration $SF = SE + EF$ is $x + (1 - x)/2 = (1 + x)/2$.

Proposition 2.2.1 Given n ropes, the set of all n-rope-measurable proper fractions are $\{k/2^n : 1 \leq k \leq 2^n - 1\}$.

Here is a proof by mathematical induction.

Proof. The proof is by mathematical induction on n. The proposition holds for $n = 1$, since $1/2$ is a 1-rope-measurable proper fraction. Note that $1/2$ is of the form "an odd number divided by 2." For $n = 2$, by Lemma 2.2.1, the two new 2-rope-measurable proper fractions are $(1-1/2)/2 = 1/4$ and $(1+1/2)/2 = 3/4$, both of which are of the form "an odd number divided by 2^2." Taking the union of sets $\{1/2\}$ and $\{1/4, 3/4\}$, we note that the proposition holds for $n = 2$.

Suppose that the proposition holds for some integer $n \geq 1$. Let there be $(n + 1)$ ropes available. Then starting from any n-rope-measurable proper fraction $k/2^n$, where k is an odd number between 1 and $2^n - 1$, in view of Lemma 2.2.1, we can measure two new proper fractions $(1 \pm k/2^n)/2 = (2^n \pm k)/2^{n+1}$. Note that $(2^n + k)$ and $(2^n - k)$ are between 1 and $2^{n+1} - 1$, and both are odd. Thus, these new proper fractions are of the form "an odd number divided by 2^{n+1}." Combining these new fractions with those fractions that are n-rope-measurable, all proper fractions of the form $k/2^{n+1}$ are $(n + 1)$-rope-measurable. Hence, the proposition holds for $(n + 1)$. This completes the proof.

Proposition 2.2.1 states that a proper fraction of the form $k/2^n$ is indeed n-rope-measurable. While the proof here is existential, in Subsection 2.5 we will give an exact construction. In Subsection 2.6, we will classify all proper fractions as either n-rope-measurable for some $n \geq 1$, or not n-rope measurable for any n. We develop some other necessary concepts in the next two subsections.

Here we prove existence. In Subsection 2.5, we will give an exact construction.

2.3 *Bisection Functions (BFs) and Their Compositions*

In view of Lemma 2.2.1, let us define two bisection functions (BFs) on $[0, 1]$, and define the composition of finitely many copies of these BFs. We study the properties of such compositions and visualize them geometrically, so that we can evaluate them on the entire domain $[0, 1]$, based on only their values at 0.

Evaluate BFs on $[0, 1]$ based on only their values at 0.

Definition 2.3.1 (Positive BF and Negative BF). The positive BF is the midpoint of (or the average of) x and 1 given by $p(x) = (1+x)/2$; and the negative BF is the "1-complement of the positive BF" given by $q(x) = 1 - p(x) = (1 - x)/2$.

Clearly, $p(0) = 1/2 = q(0)$, and $p(x) > 1/2 > q(x)$ for all $x \in (0, 1]$. Hence, the inverse function of p is well defined on $[1/2, 1]$; namely, $p^{-1}(y) = 2y - 1 \in [0, 1]$. Likewise, the inverse function of q is well defined on $[0, 1/2]$; namely, $q^{-1}(y) = 1 - 2y \in [0, 1]$.

Definition 2.3.2 (Composition of Functions). A function f of a function g is called their composition, denoted by $f \circ g$; that is, $(f \circ g)(x) = f(g(x))$ for all x.

For the BFs p and q, we show below that the composition operation is not commutative (that is, $p \circ q \neq q \circ p$); but it is linear in its argument and associative [for example, $p \circ (q \circ q) = (p \circ q) \circ q$] as shown below:

$$\begin{aligned}(p \circ q)(x) &= \frac{1 + q(x)}{2} = \frac{3 - x}{4} \\ \neq (q \circ p)(x) &= \frac{1 - p(x)}{2} = \frac{1 - x}{4}; \\ p((q \circ q)(x)) &= \frac{1 + (1 + x)/4}{2} = \frac{5 + x}{8} \\ = (p \circ q)(q(x)) &= \frac{3 - (1 - x)/2}{4} = \frac{5 + x}{8}.\end{aligned}$$

Hence, we simply write $p \circ q \circ q$, since there is no ambiguity about which composition operation is executed first. In fact, non-commutativity, linearity and associativity properties hold for higher order compositions as well. Henceforth, we denote a composition of several copies of p and q functions by the corresponding vector of BFs or by the corresponding word, always maintaining the same order of operations. For instance,

These properties hold for higher-order compositions as well.

$$(p \circ q \circ p \circ p \circ p \circ q)(x) = p(q(p(p(p(q(x))))))$$

we write either as a vector (p, q, p, p, p, q), or as a word pqp^3q, to be evaluated at $x \in [0, 1]$.

Figure 2 shows all compositions of p and q involving $n = 1, 2, 3$ letters. In Row 0, we draw an arrow on $[0, 1]$ pointing right. In Row 1, we draw the arrow **p** on $[1/2, 1]$ pointing right, and the arrow **q** on $[0, 1/2]$ pointing left. In each successive row (which need not be equally spaced), we shrink the arrows in the previous row by a factor half and shift them right by $1/2$ so that together they cover $[1/2, 1]$; and we label them by prefixing a **p** to the ordered labels in the previous row. Then we take 1-complements of these new arrows on $[1/2, 1]$ so that their 1-complement arrows (now oppositely directed) cover $[0, 1/2]$; and we label each 1-complement arrow by replacing the initial **p** in the label of its preimage by a **q**.

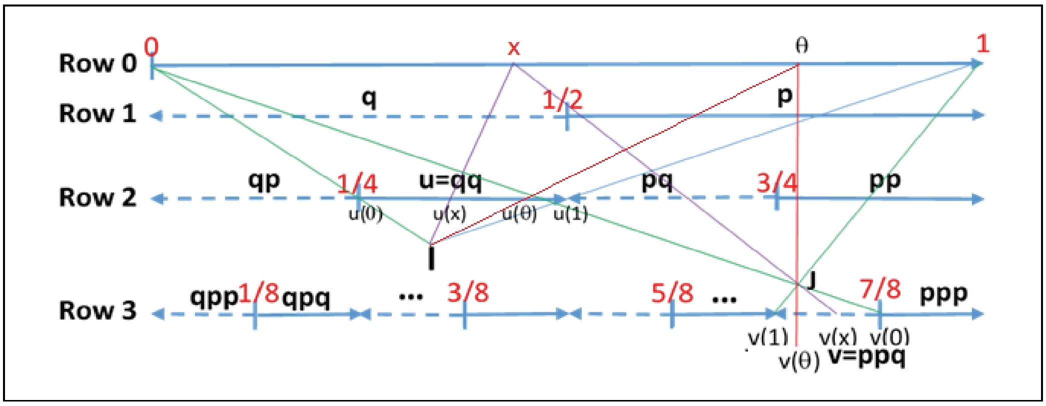

Figure 2. We show all compositions of $n = 1, 2, 3$ BFs, using a solid arrow pointing right when the composition contains even-many q's, and a dashed arrow pointing left when it contains odd-many q's. We leave you to fill in the missing labels $(\cdots)$ on four arrows.

Let v denote any composition word consisting of n letters p's and q's in some order. Then $v(0) = k/2^n$ (with k odd) is the tail of an arrow, $v(1) = k/2^n$ (with k even) is the head of that same arrow, and $v(x), x \in (0, 1)$, is a point on the same arrow obtained by linear interpolation between $v(0)$ and $v(1)$. That is, for $x \in [0, 1]$, we have

$$v(x) = v(0) + [v(1) - v(0)]\, x \tag{1}$$

For example, in *Figure* 2, we exhibit $v = ppq$, which has odd many (in fact, only one) q. Hence, the arrow points left, and $v(1) < v(0)$.

As the number of letters n increases, the arrows in Row n become progressively shorter (of length 2^{-n}); equivalently, the composition functions become progressively flatter with the slope $[v(1) - v(0)]$ in (1) having absolute value 2^{-n}. The sign of the slope is determined by the next lemma.

Lemma 2.3.1 If v is any word consisting of n letters among which q is repeated β times and p is repeated $(n - \beta)$ times in some arbitrary order, then for any $x \in [0, 1]$, we have

$$v(x) = v(0) + (-1)^{\beta}\, 2^{-n}\, x \tag{2}$$

We leave the details of the proof by induction on n to the reader, with the following hints:

$$p(v(x)) = p(v(0)) + 2^{-(n+1)}(-1)^{\beta}\, x$$
$$q(v(x)) = q(v(0)) + 2^{-(n+1)}(-1)^{\beta+1}\, x$$

In view of Lemma 2.3.1, we can interpret geometrically the functional value $v(x)$ at $x \in (0, 1)$: Suppose that v is the composition of n BFs. Let the line joining 0 in Row 0 to $v(0)$ in Row n and the line joining 1 in Row 0 to $v(1)$ in Row n meet at J. Then the line joining x in Row 0 to J intersects the arrow v at the point $v(x)$. In *Figure* 2, we exhibit $v = ppq$ and $u = qq$ to illustrate the meanings of $v(x)$ and $u(x)$. We define as I the point where the line joining 0 in Row 0 to $u(0)$ in Row m and the line joining 1 in Row 0 to $u(1)$ in Row m intersect. Moreover, in view of (2), it suffices to evaluate a composition function $v(x)$ only at zero; that is, $v(0)$. This we do in Subsection 2.4.

It suffices to evaluate the composition function only at 0.

For now, we leave you to ponder the meanings of the following two items also shown in *Figure* 2:

What does the vertical line through J mean?

[1] The points θ and $v(\theta)$, where the vertical line through J meets Row 0 and Row n respectively, so that $\theta = v(\theta)$.

[2] The point $u(\theta)$ where the oblique line joining θ to I meets Row m.

We reveal their meanings in Subsection 3.2, after Corollary 3.2.3.

2.4 *Evaluating* $v(0)$*, and Inverting the Process*

Although $p(x) > 1/2 > q(x)$ for all $x \in (0, 1]$, for $x = 0$, we have both $p(0) = 1/2$ and $q(0) = 1/2$. So, we write $b(0) = 1/2$, where b is either p or q. If u is any composition of $(n-1)$ BFs, then $up(0) = uq(0) = k/2^n$, where $1 \le k \le 2^n - 1$ is an odd number. For example, $pqpqb(0) = 19/32$ as shown below

$$0 \xrightarrow{b} \frac{1}{2} \xrightarrow{q} \frac{1}{4} \xrightarrow{p} \frac{5}{8} \xrightarrow{q} \frac{3}{16} \xrightarrow{p} \frac{19}{32} \qquad (3)$$

Conversely, given a proper fraction of the form $k/2^n$, where k is an odd number, we can obtain a word u, an ordered $(n-1)$-tuple

	BCF	p	q	p	q	q	b
numerator	37	5	11	3	1	1	0
denominator	64	32	16	8	4	2	1

Table 1. How to construct the BCF representation of 37/64.

of the p and the q functions, and augment $b(0)$ to its right. The entire vector (u, b) or word ub so found is called the bisectional continued fraction (BCF) representation of $k/2^n$. For example, 37/64 has the BCF representation given in *Table* 1.

Here is the explanation: We double the numerator 37 to get 74, which exceeds the denominator 64 by 10. Since 10/64 reduce to 5/32, we write $37/64 = p(5/32)$. Next, we double the numerator 5 to get 10, which falls short of the denominator 32 by 22. Since 22/32 reduce to 11/16, we write $5/32 = q(11/16)$. Combining the two steps so far, we write $37/64 = p(5/32) = pq(11/16)$. We continue in this manner for several more steps until we obtain $37/64 = pqpqqb(0)$.

Obtaining the BCF representation of a BF, as done in *Table* 1, is called the bisectional algorithm.

Definition 2.4.1 (Bisectional Fraction and n-Rope-Measurable Fraction). A fraction of the form $k/2^n$, where $1 \leq k \leq 2^n - 1, n \geq 1$, is called a bisectional fraction, because it has a BCF representation in finitely-many (at most n) letters. If the numerator k is odd, we call $k/2^n$ an n-rope-measurable fraction, because we can measure $k/2^n$ hour using n ropes. If k is even, we reduce $k/2^n$ to $l/2^m$, where l is odd and $m < n$. Consequently, the reduced fraction becomes an m-rope-measurable fraction.

Definition 2.4.2 (Bisectional Algorithm). Conversion of a bisectional fraction to its BCF representation (that is, a vector or word of at most n positive and negative BFs), illustrated in *Table* 1, we call the bisectional algorithm.

The bisectional algorithm gives us the following result.

Lemma 2.4.1. Every bisectional fraction $k/2^n$ corresponds to a unique composition of at most n BFs evaluated at 0.

Again, we leave details of the proof by induction on n to the interested reader, with this hint: If $k \geq 2^n$, then $k/2^{n+1} = p(k/2^n - 1)$; and if $k \leq 2^n$, then $k/2^{n+1} = q(1 - k/2^n)$.

The set of n-rope measurable fractions and the set of words of length at most n letters p or q have a one-to-one correspondence!

In view of Lemma 2.4.1, for any integer $n \geq 1$, there is a bijection (or one-to-one correspondence) between the set of bisectional fractions $\{k/2^n : 1 \leq k \leq 2^n - 1\}$ and the set of ordered vectors or words of length at most n, using letters p or q, and always ending in letter b (which is either p or q), given by

$$\{b; \quad pb, qb; \quad ppb, qpb, pqb, qqb; \quad \ldots; \quad pp\cdots b, \ldots, q\cdots qb\}.$$

The number of elements in these two sets are equal, since

$$2^n - 1 = 1 + 2^1 + 2^2 + ... + 2^{n-1}$$

as we can prove by mathematical induction on n.

2.5 *Rope Burning to Measure a Bisectional Fraction of an Hour*

How do we burn at most n ropes to measure $k/2^n$?

Having converted a bisectional fraction (of the form $k/2^n$) to a composition word of length at most n, how do we burn at most n ropes to measure $k/2^n$ of an hour? See the algorithm below.

Rope Burning Algorithm (How to transform a BCF representation of a bisectional fraction into a rope burning strategy.)

We read the BCF representation of $k/2^n$ *from right to left*. That is, we start with $b(0) = 1/2$, and we begin the solution as $\{\underline{\bar{1}}\}$. In each successive step, we proceed as follows:

The Rope Burning Algorithm tells us when to burn each rope at one or both ends, and how to measure a desired BF of an hour whose BCF we have obtained via the Bisectional Algorithm.

(1) If the next operation is p, then we do three things:

(a) We insert a new vertical line just before the closing };

(b) We insert the next rope label with an overbar before every vertical line enclosed within the braces (including the newly inserted vertical line); and

(c) We insert the same rope label with both overbar and under bar just before the closing brace.

(2) If the next operation is q, then we do (a)–(c) as in Step (1), and we also do a fourth thing:

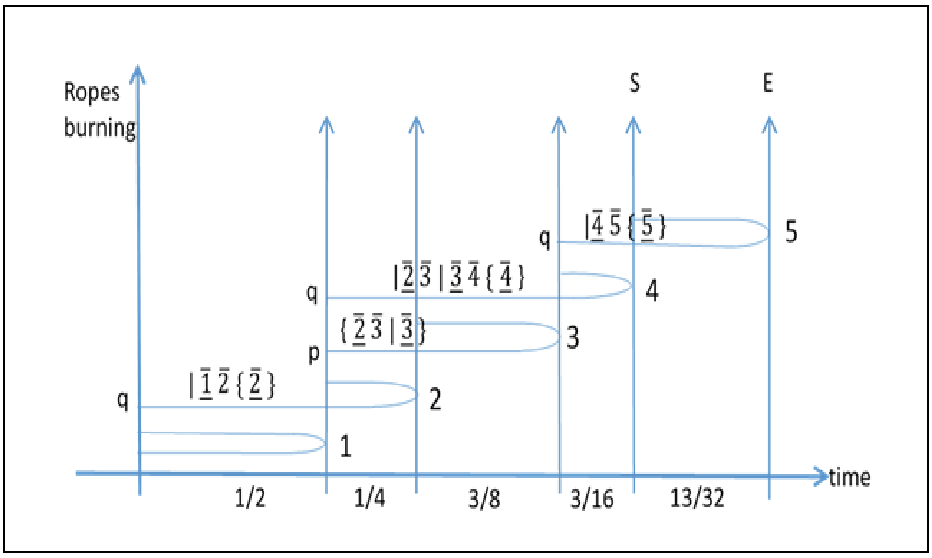

Figure 3. How to measure 13/32 of an hour? ... By using the duration when Rope 5 burns on both ends (the folded part). Here we show the burning times of the ropes, not physical points on them.

(d) We interchange the opening brace and the newly inserted vertical line so that the braces enclose only the doubly barred last rope label.

For example, the rope-burning algorithm yields

$$\frac{13}{32} = q^2 pqb(0) = |\,\underline{\bar{1}}\,\bar{2}\,|\,\underline{\bar{2}}\,\bar{3}\,\bar{4}\,|\,\underline{\bar{3}}\,\bar{4}\,|\,\underline{\bar{4}}\,\bar{5}\,\{\,\underline{\bar{5}}\,\}.$$

That is, the time duration when Rope 5 burns on both ends measures 13/32 of an hour. We depict this solution in *Figure* 3, where a folded portion of a rope means that it is burning from both ends during that time.

From *Figure* 3, we can also read off the intermediate time durations: $\{\,\underline{\bar{1}}\,\} = b(0) = 1/2$; $|\,\underline{\bar{1}}\,\bar{2}\,\{\,\underline{\bar{2}}\,\} = qb(0) = 1/4$; $|\,\underline{\bar{1}}\,\bar{2}\,\{\,\underline{\bar{2}}\,\bar{3}\,|\,\underline{\bar{3}}\,\} = pqb(0) = 1/4 + 3/8 = 5/8$; and $|\,\underline{\bar{1}}\,\bar{2}\,|\,\underline{\bar{2}}\,\bar{3}\,|\,\underline{\bar{3}}\,\bar{4}\,\{\,\underline{\bar{4}}\,\} = qpqb(0) = 3/16$.

Likewise, we measure $143/256 = pqppqppb(0)$ of an hour as

$$|\,\underline{\bar{1}}\,\bar{2}\,\bar{3}\,\bar{4}\,|\,\underline{\bar{2}}\,\bar{3}\,\bar{4}\,|\,\underline{\bar{3}}\,\bar{4}\,|\,\underline{\bar{4}}\,\bar{5}\,\bar{6}\,\bar{7}\,|\,\underline{\bar{5}}\,\bar{6}\,\bar{7}\,|\,\underline{\bar{6}}\,\bar{7}\,|\,\underline{\bar{7}}\,\bar{8}\,\{\,\underline{\bar{8}}\,\}.$$

In particular, the duration when Rope 8 burns on both ends measures 143/256 of an hour.

2.6 *For Which n is 2/3 an n-rope-Measureable Fraction?*

How to measure 2/3 of an hour?

Let us address Question Q2 posed in Subsection 2.1. To measure exactly 2/3 of an hour (or 40 minutes), will three ropes suffice? Five? Eight? ... Forty? Having determined this number, how will you burn the ropes to measure exactly 2/3 of an hour?

Try as you may, you cannot measure 2/3 of an hour exactly, even with millions of ropes! This is not your shortcoming. No one else can do it either. Whereas a measurable fraction needs only a description or a demonstration, to establish that a particular fraction is not measurable, we need a logical proof. Can you supply such a proof of impossibility?

It is impossible to measure exactly 2/3 of an hour by burning finitely-many ropes!

Lemma 2.6.1 It is impossible to measure exactly 2/3 (or 1/3) of an hour using any finite number of ropes.

Proof. As shown in Subsections 2.4 and 2.5, using n ropes we can measure $k/2^n$ hour, where $n \geq 1$ and $0 \leq k \leq 2^n$ are integers. Conversely, since there are only two permissible operations, $p(x) = (1 + x)/2$ and $q(x) = (1 - x)/2$, all n-rope-measurable fractions are necessarily of the form $k/2^n$. Since 2/3 (or 1/3) is not of the form $k/2^n$, it is impossible to measure exactly 2/3 (or 1/3) of an hour using finitely many ropes.

We can generalize Lemma 2.6.1 to prove the next result.

Most fractions are not n-rope-measurable!

Proposition 2.6.1 Given n ropes, any fraction not of the form $k/2^n$, for some $0 \leq k \leq 2^n$, is not n-rope-measurable.

However, we know all the fractions that are n-rope-measurable!

Propositions 2.2.1 and 2.6.1 give the main result of Section 2.

Theorem 2.6.1 Given n ropes, the complete set of all n-rope-measurable fractions is $\{k/2^n : 1 \leq k \leq 2^n - 1\}$.

We also can answer Question Q1 posed in Subsection 2.1. The BCF representation of 9/16 is $pqpb(0)$. Proposition 2.2.1 says that $9/16 = 9/2^4$ of an hour is 4-rope-measurable. However, since 9/16 is not of the form $k/2^3$ (for some integer k), Proposition 2.6.1 says that 9/16 is not 3-rope-measurable.

2.7 *Approximately Measuring 1/3 (or 2/3) of An Hour*

Even though by Lemma 2.6.1 above 1/3 of an hour is not n-rope-measurable for any finite n, we can measure as close to 1/3 of an hour as we want, if we have enough ropes. How?

Note that 1/3 is a fixed point of the BF q; that is, $1/3 = q(1/3)$. Applying this fixed-point property repeatedly, we have

$$\frac{1}{3} = q^n\left(\frac{1}{3}\right) = \lim_{n\to\infty} q^n\left(\frac{1}{3}\right) = q\left(\frac{1}{3}\right)$$

which involves an infinitely long word. If we truncate this word after, say, n places, we can approximate 1/3 by the terminated expression evaluated at 0, since, in view of (2)

An infinitely long word may be truncated and evaluated at 0 to produce an approximate value.

$$\left|q^n(0) - \frac{1}{3}\right| = \left|q^n(0) - q^n\left(\frac{1}{3}\right)\right| < 2^{-n}.$$

The approximation gets better and better as n increases. In fact, words $q, q^2, q^3, \ldots$, evaluated at 0, map to fractions

$$1/2, 1/4, 3/8, 5/16, 11/32, 21/64, 43/128, 85/256, \ldots \qquad (4)$$

What happens if we continue Sequence (4) *ad infinitum*?

Can you find the limit of the Sequence (4)?

In Sequence (4), the successive differences alternate in sign; and have magnitude 2^{-n}, which decrease to zero, as $n \to \infty$. Hence, Sequence (4) has a limit. Also, since every word in the sequence $(q, q^2, q^3, \ldots)$ augments one more letter q to the previous word, this limit x must satisfy $q(x) = x$; or equivalently, $(1 - x)/2 = x$; solving which, we get $x = 1/3$. Alternatively, one can prove by mathematical induction on n that the terms in (4) are

$$q^n(0) = \frac{1}{3}\left[1 - \left(-\frac{1}{2}\right)^n\right] \qquad (5)$$

Taking limit in (5), as $n \to \infty$, we get $\bar{q}(0) = 1/3$.

Thus, we can measure as close to 1/3 of an hour as we desire simply by increasing the number of ropes. We begin by burning

A letter occurring infinitely often is denoted with an overbar.

both ends of Rope 1 and one end of Rope 2. Thereafter, when one rope burns out, we ignite the other end of the burning rope, and simultaneously ignite one end of a new rope. The duration when Rope n burns from both ends gives the closest approximation to 1/3 of an hour using n ropes. Moreover, the entire duration for which Rope n burns (either from one end or from both ends) gives the closest approximation to 2/3 of an hour using n ropes. Both approximations get better as the number of ropes increases.

Approximations get better as the number of ropes increases.

We can now answer Question Q3 posed in Subsection 2.1. To measure 40 minutes (2/3 hour) within 5 seconds, we use the entire duration for which Rope 8 burns, which is 171/256 hour, and which exceeds 2/3 hour by 1/768 hour or 75/16 seconds. However, Rope 7 burns for 85/128 hour, which falls short of 2/3 hour by 75/8 seconds.

The only remaining question Q4, posed in Subsection 2.1, we will answer after proving Theorem 3.2.1.

3. An Unlimited Supply of Ropes

In Section 2, we proved that a bisectional fraction (a fraction of the form $k/2^n$) has a BCF representation having at most n letters. What about rational proper fractions with denominators other than a power of 2? Moreover, what about irrational proper fractions?

Proposition 2.6.1, in particular, says that we cannot measure a non-bisectional fraction using a finite number of ropes. Let us, therefore, assume that we have an unlimited supply of ropes, each of which burns out in exactly one hour, but at unknown, uneven and unequal rates.

Assume that we have an unlimited supply of ropes.

In Subsection 3.1, we extend the bisectional algorithm (introduced in Subsection 2.4) to construct the BCF representation of any proper fraction—rational or irrational. In Subsection 3.2, we prove that the BCF representation of a non-bisectional rational fraction is a recurrent vector or word, and vice versa. Subsection 3.3 gives the BCF and the canonical continued fraction

(CCF) representations of some familiar irrational numbers.

3.1 *Extended Bisectional Algorithm*

To obtain the BCF representation of any proper fraction—rational or irrational—we extend the bisectional algorithm.

Any fraction—rational or irrational—has a BCF representation, which can be found by using the Extended Bisectional Algorithm.

Extended Bisectional Algorithm (on $y \in (0, 1), y \neq 1/2$)

1. Write y in reduced form, if possible. Let $F = (y)$ denote an ordered reference vector of fractions already enlisted, and let $B = ()$ denote an ordered (initially empty) vector of accumulated BFs.
2. If $y > 1/2$, write $y = p(2y - 1) = p(x)$, say; and update $B = (B, p)$; that is, augment p to the right of B and redefine B. On the other hand, if $y < 1/2$, write $y = q(1-2y) = q(x)$, say; and update $B = (B, q)$. The definition of x differs in the two cases; but in both cases, $x \in (0, 1)$.
3. Check the three stopping criteria:
   (a) If $x = 1/2$, then exit the process; and update $B = (B, b)(0)$, which form the terminating BCF of y.
   (b) If $x \in F$, then exit the process. The accumulated letters in B, with the portion to the right of the first occurrence of x treated as recurring, form the recurrent BCF representation of y.
   (c) If the number of elements in B (or the number of distinct elements in F) exceeds a predetermined upper bound C, then exit the process. Declare B as the **approximate** BCF of y up to the first C components.

   If none of the stopping criteria holds, then go to Step 4.
4. Express x in reduced form, if possible; update the enlisted fractions to $F = (F, x)$; and replace y by x. Go to Step 2.

The recurrent set of letters is written with an overbar.

If the BCF is too long, truncate it after C letters.

We use the extended bisectional algorithm to construct the examples in *Tables* 2, 3 and 4 below.

Table 2. The BCF representations of some non-bisectional rational proper fractions are obtained using the extended bisectional algorithm given in Subsection 3.1

Example	rational	BCF	m	α	n	β
3.1	3/5	$\overline{pq}$	0	0	2	1
3.2	4/7	$p\overline{qpq}$	1	0	3	2
3.3	8/9	$p\overline{ppq}$	1	0	3	1
3.4	3/10	$qq\overline{qp}$	2	2	2	1
3.5	5/12	$qqp\overline{q}$	3	2	1	1
3.6	5/21	$\overline{qpqppp}$	0	0	6	2

3.2 *BCF Representation of a Non-bisectional Rational Fraction*

Suppose that we want to measure n/d of an hour, where n and d are relatively prime; and d is not a power of 2 (that is, d has at least one odd prime factor). What is the BCF representation of n/d?

For example, in Subsection 2.7, we saw that $1/3 = q(1/3) = \ldots = \bar{q}(1/3)$, and $2/3 = p(1/3) = p\bar{q}(1/3)$. Thus, 1/3 is recurrent from the start, and 2/3 is recurrent from position 2. Can we generalize these results for any non-bisectional rational fraction?

A recurrent BCF is a constant function!

We should mention that, in view of (2), any recurrent BCF is in fact a linear function of slope zero, since $\lim_{n\to\infty} 2^{-n} = 0$. In other words, a recurrent BCF is a constant function; that is, $\bar{v}(x) = \bar{v}(0)$ for all $x \in [0, 1]$. Henceforth, we shall write this constant value simply as $\bar{v}$, without specifying the argument where the function is evaluated.

In *Table* 2, we list several non-bisectional rational fractions and their BCF representations of the form $u\bar{v}$. Let m (α) denote the number of letters (q's) within u after which a recurring pattern appears, and n (β) denote the number of letters (q's) within v, which recurs.

Always, reduce fraction n/d so that n and d are relatively prime.

Theorem 3.2.1 Suppose that $n < d$ are relatively prime numbers; and d has an odd prime factor. Then the BCF representation of n/d is recurrent after m positions, where

(1) $m = 0$, if both n and d are odd;

(2) $m = 1$, if n is even and d is odd; and

(3) $m \geq 2$, if n is odd and d is divisible by 2^{m-1}, but not by 2^m.

Proof. The proof handles sequentially the three cases depending on the nature of n and d.

(1) Suppose that $n < d$ are both odd and relatively prime. Let $n_1 = n$; and define $n_2 = |2n - d|$.

(a) If $n_2 = n_1$, then $n < d$ implies that $n = d/3$. Also, since n and d are relatively prime, we have $(n = 1, d = 3)$. We already know (see Subsection 2.7) that the BCF representation of 1/3 is $\bar{q}$, for which $m = 0$.

(b) If $n_2 \neq n_1$, we note that n_2 is odd, smaller than and relatively prime to d. We repeat the process to successively define $n_3, n_4, \ldots$ etc. via $n_i = |2n_{i-1} - d|$. Note that there are finitely many possible values for n_i that are odd, smaller than and relatively prime to d. Furthermore, each n_i (including n_1) has exactly one pre-image n_{i-1} given by $(d + n_i)/2$ or $(d - n_i)/2$, whichever is odd. (Note that only one of these two values is odd, since their difference, n_i, is odd.) Therefore, the ordered sequence $(n_1, n_2, n_3, \ldots)$ passes through several odd numbers, smaller than and relatively prime to d, until it returns to n_1; that is, $n_{d^*+1} = n_1$ for some natural number d^*. (The sequence cannot return to any other n_i, with $i > 1$, since in that case n_i would have two distinct preimages: n_{i-1} and n_{d^*}.) Hence, the BCF representation of n/d is recurrent from position 1 (or $m = 0$), and v has length d^*.

There are only finitely many possible values...

The sequence cannot return to any other...

A proof by contradiction!

(2) Next, suppose that n is even, d is odd, and they are relatively prime. Define $\tilde{n} = |2n - d|$. If $2n > d$, then write $n/d = p(\tilde{n}/d)$; and if $2n < d$, then write $n/d = q(\tilde{n}/d)$. Note that $\tilde{n}$ and d are both odd and relatively prime. Hence, by Case (1), $\tilde{n}/d$ is recurrent from position 1. Consequently, n/d is recurrent from position 2.

(3) Finally, suppose that n is odd, d is even, and they are relatively prime. Let d be divisible by 2^{m-1}, but not by 2^m, for some $m \geq 2$. We shall show that n/d is recurring from position m, by mathematical induction on $m \geq 2$.

Let $m = 2$. If $2n \geq d$, we have $\frac{n}{d} = p(\frac{2n-d}{d})$. Otherwise, we have $\frac{n}{d} = q(\frac{d-2n}{d})$. In either case, $\tilde{n} = |2n - d|$ is divisible by 4, since both $2n$ and d are divisible by 2 but not by 4. Hence, gcd $(\tilde{n}, d) = 2$. We reduce $\frac{\tilde{n}}{d}$ to $\frac{\tilde{n}_1}{d_1}$, where $\tilde{n}_1 = \tilde{n}/2$ is even and $d_1 = d/2$ is odd; and $\tilde{n}_1 \leq d_1$ are relatively prime. Hence, by Case (2) above, $\frac{\tilde{N}1}{d_1}$ is recurrent from position 2; and so $\frac{n}{d}$ is recurrent from position 3 onwards.

Suppose that the theorem holds for some $m \geq 2$. Consider the next value $(M+1)$ (that is, suppose that d is divisible by 2^m, but not by 2^{m+1}). Define $\tilde{n} = |2n - d|$. Then, as before, either $n/d = p(\tilde{n}/d)$ or $n/d = q(\tilde{n}/d)$, according as $2n \geq d$ or $2n \leq d$. Since n is odd, $\tilde{n}$ is divisible by 2, but not by 4. Hence, $\gcd(\tilde{n}, d) = 2$. So, we reduce $\frac{\tilde{n}}{d}$ to $\tilde{n}_1 d_1$, where $\tilde{n}_1 = \tilde{n}/2$ is odd and $d_1 = d/2$ is divisible by 2^{m-1}, but not by 2^m; and $\tilde{n}_1 \leq d_1$ are relatively prime. By the induction hypothesis, $\tilde{n}_1/d_1$ is recurrent from position $(m + 1)$. This completes the proof by induction on m, in Case (3).

The proof of the theorem is now complete.

We are ready to answer Question Q4 posed in Subsection 2.1. To measure 35 minutes (7/12 of an hour) within one second, note that

$$7/12 = p(1/6) = pq(2/3) = pqp(1/3) = pqp\bar{q}.$$

Therefore, we first use 8 ropes to measure $q^8(0) = [1 - 2^{-8}]/3 = 85/256$ [using (5)] as the duration when Rope 8 burns on both ends. This underestimates 1/3 of an hour with an error of 1/768 hour or 75/16 seconds. Then we use three more ropes, and measure $pqp(q^8(0))$ of an hour, which in view of (2) equals

$$pqp(q^8(0)) = pqp(0) - q^8(0)/8 = \frac{5}{8} - \left(\frac{85}{256}\right)/8 = \frac{1195}{2048}$$

It overestimates 7/12 of an hour by 1/6144 hour or 75/128 second. Fewer than 11 ropes will not suffice to measure 7/12 of an hour within one second.

Fewer than 11 ropes will not suffice...

The converse of Theorem 3.2.1, given below, evaluates recurrent BCFs of the form $\bar{v}$ and $u\bar{v}$, where u and v are finite words.

Theorem 3.2.2 Let v be a word of length n, involving β copies of q and $(n-\beta)$ copies of p in some order. The terminating BCF $v(0)$ evaluates to a fraction $k/2^n$ for some integer $1 \leq k \leq 2^n - 1$; and the recurrent BCF $\bar{v}$ evaluates to

$$\frac{v(0)}{1 - 2^{-n}\,(-1)^{\beta}} = \frac{2^n v(0)}{2^n - (-1)^{\beta}} = \frac{k}{2^n - (-1)^{\beta}} \tag{6}$$

One may replace the right hand side by its reduced form.

Proof. We have already seen in (3) how a terminating BCF v of length n, when evaluated at 0, results in a bisectional proper fraction of the form $k/2^n$ for some $1 \leq k \leq 2^n - 1$.

To evaluate $\bar{v}$, first let $\bar{v} = w$. Then note that $v(w) = v(\bar{v}) = \bar{v} = w$; that is, w is a fixed point of v. Next, from (2), we have

$$w = v(w) = v(0) + 2^{-n}(-1)^{\beta} w$$

which has a solution in w, given in (6). This completes the proof.

Corollary 3.2.3 Let u and v be two finite words of lengths m and n respectively. Suppose that $u(0) = l/2^m$ and $v(0) = k/2^n$ for some integers $1 \leq l \leq 2^m - 1$ and $1 \leq k \leq 2^n - 1$. Also, let α denote the number of q in u (in any order) and β denote the number of q in v (in any order). Using (2) and Theorem 3.2.2, the BCF $u\bar{v}$ evaluates to

The vertical line through J satisfies the fixed-point property; and hence, it has a recurrent BCF representation.

$$u(\bar{v}) = u\left(\frac{k}{2^n - (-1)^{\beta}}\right) = \frac{l}{2^m} + \frac{(-1)^{\alpha}}{2^m}\,\frac{k}{2^n - (-1)^{\beta}} \tag{7}$$

For a geometric interpretation of $\bar{v}$ and $u\bar{v}$, let us return to *Figure* 2. The vertical line through J, meeting Row 0 at θ and Row n at $v(\theta)$, satisfies the fixed-point property: $v(\theta) = \theta$. Therefore, $\theta = \bar{v}$. Likewise, the point $u(\theta)$, where the oblique line joining θ to I intersects the arrow u in Row m, in fact, represents $u(\theta) = u\bar{v}$.

Table 3. Some recurrent BCF representations are evaluated using Equations (6) and (7) to obtain rational fractions.

Example	BCF	m	α	n	β	rational
3.7	$\overline{qpq}$	0	0	3	2	1/7
3.8	$\overline{qqqpp}$	0	0	5	3	3/11
3.9	$\overline{qqpqqq}$	0	0	6	5	27/65
3.10	$qqpp\overline{q}$	4	2	1	1	11/24
3.11	$\overline{qqqqqppqqp}$	0	0	10	7	13/41
3.12	$pq\overline{qqqqqppqqp}$	2	1	10	7	55/82

Theorem 3.2.2 and Corollary 3.2.3 may give the impression that all recurrent words correspond to fractions whose denominators factor into a power of 2 times a power of 2 plus or minus 1 (depending on whether β is odd or even). However, since we may reduce the right hand side of (6), the denominator need not be of these forms only. In fact, it can be arbitrary as the examples in Table 3 illustrate and as Corollary 3.2.4 proves.

Corollary 3.2.4 Any odd number d divides some power of two plus or minus one; that is, d divides a number of the form $(2^n \pm 1)$, for some $n \geq 1$.

Proof. Since d is odd, by Theorem 3.2.1, the fraction $1/d$ has a recurring BCF representation of the form $\bar{v}$ (that is, $1/d$ is recurrent starting from position 1). Next, by Theorem 3.2.2, $\bar{v}$ evaluates to $k/[2^n - (-1)^\beta]$, which should equal $1/d$. Hence, $2^n - (-1)^\beta = kd$; or d divides $2^n - (-1)^\beta$. This completes the proof.

To illustrate the above proof of Corollary 3.2.4, let $d = 41$. Then by Theorem 3.2.1, $1/41 = \overline{qppppqpqpp}$. Hence, $m = 0, \alpha = 0, n = 10, \beta = 3$. Next, note that $qppppqpqpp(0) = 25/1024$. Therefore, by Theorem 3.2.2, $\overline{qppppqpqpp}$ evaluates to $\frac{2^{10}*25/1024}{2^{10}-(-1)^3} = \frac{25}{1025} = \frac{1}{41}$. In other words, 41 divides $1025 = 2^{10} + 1$.

The statement of Corollary 3.2.4 is independent of Theorems 3.2.1 and 3.2.2. This may motivate one to find a proof that does not invoke Theorems 3.2.1 and 3.2.2. Below we give such a proof.

Proof of Corollary 3.2.4 (not using Theorems 3.2.1 and 3.2.2). Since d is odd, it is relatively prime to 2. Surely, the beginning odd numbers 1, 3, 5, 7, 9 are already of the desired form. For $d \geq 11$, let $a_i = 2^i \pmod{d}$, for $i \geq 1$. Clearly, $0 \leq a_i \leq d - 1$, for

$i \geq 1$. Therefore, there exist $1 \leq i < j$ such that $a_j = a_i$. This, in particular, implies that d must divide $2^j - 2^i = 2^i(2^{j-i} - 1)$. But, since d is odd, it must divide $(2^{j-i} - 1)$, which can be factored further whenever $(j - i)$ is even. This completes the proof.

To illustrate this alternative proof of Corollary 3.2.4, let $d = 21$. Then the sequence $(a_i; i \geq 1)$ becomes $(2, 4, 8, 16, 32 = 11, 22 = 1, 2, \ldots)$. Thus, $a_7 = a_1$. Hence, 21 divides $2^6 - 1$. This explains why $n = 6$ in Example 3.6.

3.3 *BCF and CCF Representations of Irrational Fractions*

We apply the extended bisectional algorithm of Subsection 3.1 on to some (upto 20 letters) familiar irrational numbers, and document their BCF representations in *Table* 4 below. By way of comparison, we (upto 11 places) also obtain the canonical continued fraction (CCF) representations of those same irrational numbers. Recall that the CCF is of the form "an integer plus a proper fraction, where one writes the fractional part as the reciprocal of another CCF." As a shorthand, we write the successive integers within square brackets. See [3]. For example,

$$\frac{138}{61} = 2 + \frac{1}{61/16} = \ldots = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \frac{1}{3}}}}$$

we write as $2 + [3, 1, 4, 3]$. Alternatively, $2 + [3, 1, 4, 2, 1]$ would also do; but it is the shorter representation which is known as canonical. Every rational number has a CCF representation, which terminates after a finite number of steps. Conversely, an irrational number has a CCF representation involving an infinite number of steps that may exhibit a recurring pattern, a discernible pattern though not recurring, or no pattern at all.

In contrast to CCF, by Theorem 2.6.1, only a bisectional rational number has a BCF representation involving a finite number of letters. By the results of Subsection 3.2, every non-bisectional rational number has a recurrent BCF representation and vice versa. Therefore, the BCF representation of an irrational number cannot exhibit a recurring pattern. Neither did we find a discernible

Irrational Fraction	BCF [CCF]	Rational Approximation Numerator	Denominator
$\sqrt{2}-1$	$qqpqqqqqppppqqpqpppq\cdots$	7192	17363
	$[2,2,2,2,2,2,2,2,2,2,2,\cdots]$	2378	5741
$\sqrt{3}-1$	$pqqppqqpqqpqpqpppqqq\cdots$	418	571
	$[1,2,1,2,1,2,1,2,1,2,1,\cdots]$	418	571
$\phi-1=$	$pqpqpppqppqpqqppqqpp\cdots$	55	89
$(\sqrt{5}-1)/2$	$[1,1,1,1,1,1,1,1,1,1,1,\cdots]$	89	144
$e-2=$	$pqqpqqpppppqpppqqqqq\cdots$	719	1001
$.718281828\cdots$	$[1,2,1,1,4,1,1,6,1,1,8,\cdots]$	719	1001
$\pi-3=$	$qpqqpqqpppqpppppqqpq\cdots$	148471	1048576
$.141592653\cdots$	$[7,15,1,292,1,1,1,2,1,3,1,\cdots]$	192583	1360120
$\sqrt{\pi}-1=$	$ppqppqqqpqqppppppqpp\cdots$	39097	50614
$.772453851\cdots$	$[1,3,2,1,1,6,1,28,13,1,1,\cdots]$	52265	67661
$\pi^e-22=$	$qqppqqqqpqppqqqpqqqq\cdots$	697	1518
$.459157718\cdots$	$[2,5,1,1,1,1,1,3,2,1,1,\cdots]$	905	1971
$e^\pi-23=$	$qpqqpqqppppppqqppqpq\cdots$	147527	1048576
$.140692632\cdots$	$[7,9,3,1,1,591,2,9,2,44,1,\cdots]$	68461033	486599985

Table 4. Using extended bisectional algorithm, we document the BCF representations (up to 20 letters) of some irrational proper fractions, their CCF representations (up to 11 terms), and rational values of these approximate representations.

pattern among the BCF representations of the few irrational numbers we studied. We invite readers to find a familiar irrational number whose BCF representation exhibits a discernible pattern. Alternatively, the reader may try to evaluate a number whose BCF representation has a non-recurrent, discernible pattern. For example, evaluate $pqpqqpqqqpqqqqp\cdots$, where each p is followed by one more q than in the previous instance.

4. Concluding Remarks

Write your own codes, or contact the author for R codes.

Many of the examples in this paper we have constructed manually; but, of course, not all. For longer examples and for Table 4, we have used the freeware R (see [4]). We urge readers to construct other examples. They may write their own codes in their favorite language, or contact the author for R codes.

Here we conducted an extended intellectual tour, which started with an interview question, continued on to a generalization, and culminated in several theorems. We hope the journey has been both pleasant and profitable to you. We left one problem unsolved: Find an (otherwise familiar) irrational number whose BCF representation (though not recurrent) does exhibit a discernible pattern.

We encourage readers to solve mathematical problems, incorporating thoughtful reflections. A good source of many easy-to-understand problems is [5]. Happy hunting.

Acknowledgment.

I thank Professor B. Sury, who introduced the rope-burning puzzle to me; and then nudged me on to swallow the pill I had prescribed to others. I also thank my colleagues, Professors Michal Misiurewicz and Patrick Morton, for some discussions. I am indebted to an anonymous referee for numerous suggestions for improving an earlier manuscript, including the two proofs of Corollary 3.2.4. I dedicate this paper to Professor George Gopen, whose Scientific Writing workshop and book [6] affected my writing style.

Suggested Reading

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