Classroom

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Meet Your Match*

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The author did not learn probability theory properly when it was taught to him as an undergraduate. However, now that he has to *teach* it has become a fascination. The following problem is based on a section in the book *Asymptopia* by Joel Spencer and Laura Florescu. The original problem had a "life or death" formulation. This version is less murderous!

We have 52 gamblers who are given the following game.

- First of all, each gambler is given one card from one pack (say with a Blue back) of 52 distinct cards.
- There is a room with a table. The game master arranges 52 cards from another pack of cards (say with a green back) into 4 rows of 13 cards each. (The cards have been shuffled, and put face down, and the game master has noted the exact position of each card.)
- Each gambler must go into the room and look at the face of at most 26 Green cards. If the gambler finds the matching green card to the blue card she/he already has, then she/he has won his/her round.

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What is the best strategy? The gamblers should use group theory!

- After the gambler leaves the room, the green cards are put back *exactly* as before she/he came in. Moreover, the gambler cannot meet the other gamblers after leaving the room and take the blue card away.
- The gamblers are allowed to come up with a common strategy at the beginning.

The question is: "What is the best strategy for the gamblers?" "Do the chances of winning change if we only have 40 gamblers and leave out all the 'picture' cards?" "What are the optimal probabilities?"

1. Some Hints:

- If the gamblers do not employ any strategy and each one looks at 26 cards at random, then each has 1/2 chance of winning his round.
- The gamblers should use group theory!

2. Mathematical Formulation

Basically, the problem can be posed as follows:

- There is a "permutation" map $\sigma : [1, 2n] \to [1, 2n]$; this map is one-to-one and onto. (Here $2n = 52$.) This permutation is *fixed* throughout the game.
- Each gambler is numbered i in the range $[1, 2n]$ and examines $\sigma(j)$ for some n values of j. If the gambler finds a j such that $\sigma(j) = i$, then the gambler wins.
- The strategy followed by one gambler cannot depend on the information gathered by the other gamblers.

The problem is to devise a common strategy for each gambler which increases the probability of all gamblers winning.

The strategy below is based on the (reasonable) assumption that *all* permutations σ are equally likely.

3. Break Out into Cycles

The permutation σ creates a *partition* of the set $[1, 2n]$ into "cycles" of the form $C = (i, \sigma(i), \sigma^2(i), \dots)$. These cycles are finite and the last element of the cycle C is sent to i by σ; in fact, you get the same cycle starting with any element of it.

The gambler "follows the cycle". If his cycle has length at most n then he wins his round.

Suppose that a gambler "follows the cycle". In other words, he starts with opening the box that has his number on the outside, then opens the box numbered with the number he finds inside the box and so on. If his cycle has length *less* than or equal to n, then he wins his round since at the end he finds the number of the box he started with—but that that is his own number!

So how many cycles can there be of length greater than n? At most one! All the gamblers who are *not* in that one cycle will win if they follow the above strategy. Can it happen that *all* the gamblers are in that one cycle? Sure. We could have a single cycle of length $2n$.

The shuffling of the cards (in other words, the choice of σ) was assumed to be "random". So, to count our losses, we need to count for each $k \geq 1$, the number of permutations that contain an $n + k$ cycle among the collection of *all* permutations. The collection of all permutations is of size $2n!$. The permutations that contain an $n + k$ cycle can be counted as follows. Let us first count those for which the $n + k$ cycle consists of the some chosen $n + k$ elements; there are $(n + k - 1)!$ cycles that can be made out of such a chosen set. The rest of the permutation is then a permutation of the remaining $n - k$ elements; there are $(n - k)!$ choices for this. Thus, the number of permutations where some chosen $n + k$ elements are in a cycle is $(n - k)!(n + k - 1)!$. Since we could have chosen these $n + k$ elements in $\frac{(2n!)}{(n-k)!(n+k)!}$ ways, we see that the number of permutations with an $n + k$ cycle is $(2n)!/(n + k)$.

So the probability of at least one gambler losing with this strategy is $p_n = \sum_{k=1}^{n} 1/(n+k)$ which increases to $\log(2)$ (or approximately 0.69) as n goes to infinity.

There is about 69% chance of at least one gambler losing. On the other hand, there is a 31% chance that they will all win!

In other words, even if there are a large number of gamblers, there is about 69% chance of at least one gambler losing. On the other hand, for large n, there is a 31% chance that they will all win!

Note that if each gambler adopts the "random choice" strategy, then the probability that they will all win is $1/2^{2n}$, which is much smaller. However, this is also the probability that they will all lose! On the other hand, with the above strategy, the probability that they will all lose is $1/(2n)$ which is much larger! So, if the objective is to avoid a total loss, then the random strategy is better than the above strategy!

What is the expected number of successes? With the random choice strategy this is $n = 2n \cdot (1/2)$. On the other hand, with the above strategy, note that for a permutation that does not contain a cycle of length $> n$, there are $2n$ wins, while if there is a $n + k$ cycle, then the remaining $n - k$ win. Thus, the expected number of wins is

$$2n\left(1 - \sum_{k=1}^{n} \frac{1}{n+k}\right) + \sum_{k=1}^{n} \frac{n-k}{n+k} = 2n - \sum_{k=1}^{n} \frac{2n-(n-k)}{n+k} = n\,.$$

In other words, the expected number of wins is unchanged!

What is the variance around this expected number? For the random strategy, this is $n/2 = 2n \cdot (1/2)^2$. For the above strategy, by the same reasoning as above it is

$$n^2\left(1 - \sum_{k=1}^{n} \frac{1}{n+k}\right) + \sum_{k=1}^{n} \frac{k^2}{n+k} = n^2 - \sum_{k=1}^{n} \frac{(n^2-k^2)}{n+k} = n(n-1)/2\,.$$

This is much larger for large n.

Also note that for an individual gambler, the probability of losing is given by adding the probability that she/he is in a cycle of length $n+k$ given that there is a cycle of length $n+k$. The conditional probability that the gambler is in a chosen set of size $n+k$ is $(n+k)/2n$. It follows that the probability of losing is

$$\sum_{k=1}^{n} \frac{n+k}{2n} \cdot \frac{1}{n+k} = n \cdot \frac{1}{2n} = 1/2\,.$$

In other words, the gambler does not decrease (or increase!) her/his chances of winning her/his round by following this strategy!

No risk, no gain. "A" "random" strategy is a "safe" strategy.

The above calculation can be interpreted as "no risk, no gain". The "random choice" strategy is a "safe" strategy; a number close to n of the gamblers can be expected to win. The above, permutation-based strategy, is more risky. While the expected number of wins is unchanged, there is a much higher chance of deviation from this—on the positive side as well as on the negative side! However, given that there is a 31% chance for all of them to win, I think most gamblers would choose this strategy!