The Sounds of Music : Science of Musical Scales *

II : Western Classical Music

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A set of basic notes, or 'scale', forms the basis of music. Scales are specific to specific genre of music. In this second article of the series, we explore the development of various scales associated with Western classical music, arguably the most influential genre of music of the present time.

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Introduction

Just like an alphabet is the basis of a language, a set of basic notes known as a ***scale*** forms the basis of music. The term 'scale' comes from the Latin word '*scala*' meaning 'ladder'. Thus a scale is a ladder or a set of notes ordered according to their frequency or *pitch*. A scale ordered by increasing pitch is an ascending scale, and a scale ordered by decreasing pitch is a descending scale. Scales, in general, may or may not contain the same pitches while ascending as well as when descending. Moreover, even all of the scale steps may not be equal. Indeed, most of the traditional music scales began with unequal scale steps. Due to the principle of octave equivalence, scales are generally considered to span a single octave, with higher or lower octaves simply repeating the pattern. A musical scale represents a division of the octave space into a specific number of scale steps, a scale step being the interval between two successive notes of the scale.

There exist a number of scales depending on the number of primary notes (or fundamental frequencies) available per octave. However, the traditional '*diatonic*' (in Greek, 'diatonic' means 'progressing through tones') scale used in Western classical mu-

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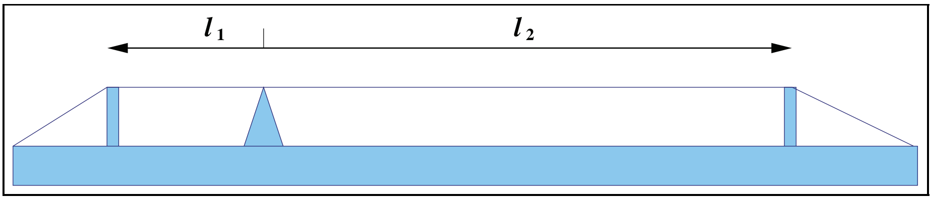

Figure 1. A Pythagorean mono-chord – A single string (chord) of length L is divided by a movable bridge into two segments of length l_1 and l_2 such that $l_1 + l_2 = L$.

sic is heptatonic (7 notes per octave). Evidently, in a heptatonic scale, doubling of frequency requires going up by 8 notes. The name octave, for a non-repetitive frequency span, derives from this. In the following we shall discuss the origin of the heptatonic scale, starting from the ***Pythagorean scale*** (one of the first theoretical tuning structures known) and its subsequent transformation into the modern ***Equal Tempered Scale*** (ETS), with 12 notes per octave, where all the notes are equidistant (in a logarithmic sense) from their neighbouring ones. In this section, we shall try to understand the mathematical logic behind these scales.

1. Pythagorean Scale

Pythagoras (*circa* 500 BC), the Greek mathematician and philosopher, used a *mono-chord* to study a) the relationship between the string length and the note produced when it is plucked (see Section 1.1, article I) and b) study the phenomenon of consonance and dissonance. A mono-chord, as seen in *Figure* 1, is a single stringed instrument with a movable bridge, dividing the string of length L into two segments, l_1 and $l_2\,(= L - l_1)$. Thus, the two string segments can have any desired ratio, $x = l_1/l_2$.

It is evident that the string tension T and the mass per unit string length μ are the same in both the string segments. When the mono-chord is played, both string segments vibrate simultaneously. Now, according to Eq.[8] of article-I, when the string is plucked, the frequencies of the fundamental vibrations generated in the two segments are,

$$\nu_1 = \frac{1}{2l_1}\sqrt{\frac{T}{\mu}}, \quad \nu_2 = \frac{1}{2l_2}\sqrt{\frac{T}{\mu}}\,. \tag{1}$$

Therefore, the ratio of these vibrations would be:

$$\frac{\nu_2}{\nu_1} = \frac{l_1}{l_2} = x\,. \tag{2}$$

It has been found that for $x = 1, 2, 3/2, 2, 4/3, 5/4, 6/5..$ the sound combination appears pleasing, in other words *consonance* occurs. These ratios actually correspond to the – unison, octave, fifth, fourth, major third, minor third (some of the most pleasing combinations) used in the Western classical music. Dissonance occurs when x is such that it can not be expressed as a ratio between two small integers.

It is believed that a tetra-chord (a four-stringed instrument) was used to generate the first set of tuning (or fixing the scale). Consider a tetra-chord with 4 strings of length l_1, l_2, l_3, l_4 with corresponding fundamental frequencies ν_1, ν_2, ν_3, ν_4. The two outer strings differs in length by a factor of 4/3 and so the frequencies differ by a factor of 4/3 as well. The middle two strings were tuned so that the ratio of the first to second strings (the first interval) was approximately the same as the ratio of the second to the third string. Therefore, we have

$$l_1 : l_4 = 4/3\,, \tag{3}$$
$$l_1 : l_2 = l_2 : l_3\,. \tag{4}$$

Assuming all the strings to have the same tension (T) and the same mass per unit length (μ) (using Eq.[8], article-I) we obtain,

$$\nu_1 : \nu_4 = 3/4\,, \tag{5}$$
$$\nu_1 : \nu_2 = \nu_2 : \nu_3\,, \tag{6}$$
$$\Rightarrow \nu_3 : \nu_4 = (\nu_2 : \nu_1)^2 \times \frac{4}{3}\,. \tag{7}$$

When $\nu_1 : \nu_2 = 9/8$, we obtain $\nu_3 : \nu_4 = 265/243$. This choice is known as the *diatonic* tuning and is the precursor of the modern Western scale. Now, let us try to understand how this choice comes about.

Pythagoras is supposed to have used the two fundamental principles of psycho-acoustics – the octave equivalence, and the consonance of the fifth. Using these two principles, we can build up a set of basic notes within an octave which would sound 'consonant' when played together. Starting with a base note (say, of frequency ν), we obtain the next one by multiplying it by 3/2 (a 'perfect fifth up' according to musical terminology), then obtain the next one by multiplying the second one by 3/2. So we now have three consonant frequencies – $\nu_1 = \nu$, $\nu_2 = 3\nu/2$, $\nu_3 = 9\nu/4$. But $\nu_3 > 2\nu_1$. This means that we have gone beyond the first octave based at ν_1. To obtain the note corresponding to the starting octave, we divide ν_3 by 2 to obtain the new $\nu_3 = 9\nu/8$ (because of octave equivalence). We can now continue the process of multiplying by 3/2 and dropping down to the correct octave to obtain the following set of notes,

$$\nu, \frac{3\nu}{2}, \frac{9\nu}{4} \sim \frac{9\nu}{8}, \frac{27\nu}{16}, \frac{81\nu}{32} \sim \frac{81\nu}{64}, \frac{243\nu}{128}, \frac{729\nu}{256} \sim \frac{729\nu}{512} ... \quad (8)$$

Collecting the frequencies within the first octave and ordering them in an ascending order, we obtain:

$$\nu, \frac{9\nu}{8}, \frac{81\nu}{64}, \frac{729\nu}{512}, \frac{3\nu}{2}, \frac{27\nu}{16}, \frac{243\nu}{128}, 2\nu, \quad (9)$$

where the note at double the frequency has been added artificially. Note that the fifth term in this series is exactly 3/2 times the base frequency, and therein lies the explanation for the name 'perfect fifth'[1] The ratios of the consecutive notes then appear in the following order:

[1] In Indian music too, the fifth note is special. We shall see later how the note called *pancham* (literally meaning the 'fifth') has been given special status in Indian tradition.

$$\frac{9}{8} : \frac{9}{8} : \frac{9}{8} : \frac{256}{243} : \frac{9}{8} : \frac{9}{8} : \frac{256}{243}. \quad (10)$$

Therefore, the adjacent notes differ either by a ratio of 9/8 (known as a Pythagorean 'tone') or by a ratio of 256/243 (known as a Pythagorean 'semi-tone'). As $(256/243)^2 \simeq 9/8$, the nomenclature is more or less justified. Pythagoras is supposed to have used these two ratios or musical intervals in his diatonic tetra-chord. Therefore, Pythagorean tuning of a tetra-chord (with respect to the base note ν) would be the following:

$$\frac{4\nu}{3}, \frac{32\nu}{27}, \frac{256\nu}{243}, \nu. \quad (11)$$

It is to be noted that if the pattern of scale-steps given by (10) is repeated successively, like the following:

$$\cdots\frac{256}{243},\frac{9}{8},\frac{9}{8},\frac{256}{243},\frac{9}{8},\frac{9}{8},\frac{9}{8},\frac{256}{243},\frac{9}{8},\frac{9}{8},\frac{256}{243},\frac{9}{8},\frac{9}{8},\frac{9}{8},\frac{256}{243}\cdots, \tag{12}$$

then at every eighth step, the frequency (approximately) doubles. This is the reason for the term *octave* to be associated with the doubling of a frequency. If 'T' (for 'tone') is used to symbolise 9/8 and 'S' (for 'semi-tone') to symbolise 256/243, the pattern obtained would be – T T S T T T S... This is the same as the pattern of intervals between the white notes on a piano – a T means there is a black note between the two white notes and S means there is no black note in between. This pattern defines the scales as well as the notes in the Western tradition. To obtain a scale, a base or home note needs to be specified. Hence, up to octave equivalence, there are seven notes to choose from, giving rise to the following seven patterns:

T T S T T T S	[C]	(major)
T S T T T S T	[D]	
S T T T S T T	[E]	
T T T S T T S	[F]	
T T S T T S T	[G]	
T S T T S T T	[A]	(minor)
S T T S T T T	[B]	

The first of these is obtained if one starts at a "C" on the piano keyboard. Historically, all of these scales were used, but over time, only the first (major scale) and the sixth (minor scale) survived[2].

[2] It should be noted that the Indian *saptak* is actually a ***major*** scale, allowing for the base note to have any frequency at all.

Even though this method of setting up a musical scale has mostly been attributed to Pythagoras, there have also been others (for example, Eratosthenes) responsible for developing the scale. Pythagorean tuning is well-suited to music that emphasises musical fifths and octaves and has been used almost exclusively by European musicians till about 16th century. Unfortunately, intervals other than the octave and the fifth are not quite perfect in this tuning. In particular, the major third (81/64) and minor third (32/27) are rather dissonant. Moreover, there is very little freedom to modulate from the home key to more distant keys. Because, going up a fifth a certain number of times never equals going up by a number of octaves. As musical thirds (intervals defined by frequency ratios of 6/5, 5/4) became more important to musical expressions, and as the desire to move far beyond the home key grew, other alternative temperaments were developed later on.

2. Equal Tempered Scale

It has been seen that the Pythagorean scale has basically been generated by taking a base note and multiplying it by 3/2 again and again, modulo a factor of 2 (to account for the octave equivalence).

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$$(3/2)^{12} = 129.746 \simeq 1.013 \times 2^7 \,. \tag{13}$$

In other words, it takes 12 steps around what is called the 'circle of fifths' until an approximate power of two (an integral number of octaves above) of the original note is reached. Therefore, the simplest explanation for the twelve tones - C, C#, D, D#, E, F, F#, G, G#, A, A#, B – (# denotes a sharp note) - within an octave is the above relation between 3/2 and 2. However, as $(3/2)^{12}/2^7$ is not exactly equal to 1, powers of 3/2 (or 'circles of fifths') would not take us exactly to twice the original frequency[3]

[3]Because, the 'circle of fifth' does not quite close in Pythagorean tuning, one of the intervals must *NOT* match the prescribed frequency ratio, in order to close the circle forcibly. Then there would be a dissonant beat. This is known as a *'wolf interval'*(meaning the interval howls like a wolf!).

In an attempt to maximise the number of consonant intervals having exact frequency ratios within the octave the *'Just Scale'* (also known as the 'harmonic tuning' or 'Helmholtz's scale') was created. This scale starts from a perfect triad – a base note (ν), the perfect third ($4\nu/3$) and a perfect fifth ($3\nu/2$) – which has a frequency ratio of 4:5:6. The construction of the rest of the scale is as before. One goes up from a given note by a major third ($\nu \to 4\nu/3$) or by a perfect fifth ($\nu \to 3\nu/2$), and the resulting note is brought back to the home octave by descending an appropriate number of octaves ($\nu \to \nu/2^n$) till the note having twice (or approximately twice) the base note is achieved.

However, the Just Scale too suffers from problems similar to those encountered by the Pythagorean scale. It appears to be impossible to create a tuning system that has perfect intonation for all the consonant musical intervals within the octave. There appears to be two basic problems:
a) to have the perfect intonation (correct ratio of frequencies) for all consonant musical intervals within the octave,
b) to have the freedom to modulate, to move away from the base octave without losing consonance.

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The ETS, created around the 19th century, specifically addresses these particular problems. Since there are 12 notes, the octave is divided equally (in a logarithmic sense) in 12 steps and each note is obtained by multiplying the previous one by $2^{\frac{1}{12}}(\sim 1.0595)$. The frequency ratios from the base note in this scale are as follows – 1.0000, 1.0595, 1.1225, 1.1892, 1.2599, 1.3348, 1.4142, 1.4983, 1.5874, 1.6818, 1.7818, 1.8877, 2.0000. Evidently, all octaves would be perfect now, with a ratio 2:1. Therefore, there is absolute freedom to modulate from one octave to another as there is no distinction between them and they all contain the same semitone intervals, and have the same values for all the other (slightly compromised) intervals as well. Thus, the actual frequencies obtained are slightly different from the Pythagorean scale.

Notes (major scale)	C4	E4	G4
Pythagorean (Hz)	260.741	330.00	391.11
ETS (Hz)	261.625	329.63	392.00

However, the difference between the frequencies obtained using a Pythagorean scale and the ones obtained in ETS are rather small. Assuming *A* to be at 440 Hz (concert tuning)[4], the frequencies of the notes of a major chord are as seen above. It is evident that the differences are too small to be perceptible to the human ear. This is the reason why, at the present time, the keys of most reed instruments (the piano, the electronic keyboard or the harmonium,[5]) are tuned according to the ETS.

[4]The absolute pitch, or base key, in the baroque period used to be tuned at A = 415 Hz, which was used by Bach while Handel used 422.5 Hz. The international tuning standard of A4 = 440 Hz was widely adopted around 1920.

There is an interesting way to see the connection between the ETS

[5]Harmonium – Originally an European reed instrument, a modified pump organ, was introduced to Asia by Christian missionaries. It gained huge popularity in Asia as an accompaniment and was modified to suit the Asian musician's practice of sitting on the floor.

and the consonance of fifth. Basically, ETS is being defined by the relation $(3/2)^m \simeq 2^n$ where both m, n are integers (Of course, the exact equality can never hold true.). Therefore, we are trying to represent a real number x by the quotient of two (not very large) integers m and n where x is given by:

$$3/2 = 2^x \Rightarrow x = \frac{\log 3/2}{\log 2} = 0.584962500721... \tag{14}$$

Now, any real number can be expressed in terms of a continued fraction where the expansion has 1 in all the numerators and continues forever. The reason for using continued fractions is that these give the best rational approximations to real numbers, i.e. any closer approximation would have a larger denominator. When x is expressed thus, we obtain:

$$\frac{\log 3/2}{\log 2} = \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{3+\frac{1}{1+\frac{1}{5+\frac{1}{2+\frac{1}{23+\frac{1}{2+..}}}}}}}}}} \tag{15}$$

Taking the first few terms as approximate values for this continued fraction, we obtain:

$$\text{Approximation 1.} \quad \frac{1}{1+\frac{1}{1}} = \frac{1}{2} = 0.5 \tag{16}$$

$$\text{Approximation 2.} \quad \frac{1}{1+\frac{1}{1+\frac{1}{2}}} = \frac{3}{5} = 0.6 \tag{17}$$

$$\text{Approximation 3.} \quad \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2}}}} = \frac{7}{12} = 0.58\dot{3} \tag{18}$$

$$\text{Approximation 4.} \quad \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{3}}}}} = \frac{24}{41} = 0.5854.. \tag{19}$$

The approximation improves when more number of terms are included. However, when more terms are included, the fraction is obtained in terms of larger and larger integers. Evidently, the ratio

7/12 appears to be an excellent compromise which immediately suggests an octave of 12 steps. Moreover, it should be noted that $2^{\frac{7}{12}} = 1.498.. \simeq 3/2$, which shows that a fifth of the Pythagorean is equal to 7 steps of this equal tempered scale.

With time ETS has all but replaced the original Pythagorean scale. In particular, most reed instruments are tuned to this scale. Of course, musicians using string instruments and vocalists still have the freedom to choose a scale according to their convenience. One of the traditions where this freedom has been of paramount importance is the Indian classical music. In the next and final article of this series, we shall discuss the scales used in that genre.

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Suggested Reading

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