Win or Lose? Listen to a Martingale!*

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The author did not learn probability theory properly when it was taught to him as an undergraduate. However, now that he has to *teach*, it has become a fascination. The problem stated below was given to second year students of the IISER BS-MS programme as an examination question; they answered it in interesting ways. An earlier article in *Resonance* [1] also talks about martingales.

We play various games of dice and use probability theory to calculate our expected winnings.

1. The Simplest Case

A fair dice (6 sides numbered 1 to 6) is tossed repeatedly. You have 5 rupees in your pocket, and you have to give away 1 rupee each time a number different from '6' shows.

A fair dice is tossed repeatedly. You have 5 rupees in your pocket, and you have to give away 1 rupee each time a number different from '6' shows. What is the expected number of times you have to throw the dice before your pocket is empty?

Question: What is the expected number of times you have to throw the dice before your pocket is empty?

The 'standard' solution of this simple problem is to define a random variable W as the number of '6's is thrown before a non-six is thrown at least 5 times (not necessarily in sequence). In that case, W follows the negative binomial distribution:

$$P(W = k) = \binom{k + 5 - 1}{5 - 1} \frac{5^5}{6^{5+k}}$$

The total number of throws is $W + 5$ before the game stops (since

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your pocket is empty!). Its (mathematical) expectation is

$$E(W+5) = E(W) + 5 = \frac{5 \cdot (1/6)}{5/6} + 5 = 6$$

2. More Complicated Case

A fair dice is tossed repeatedly. You have 5 rupees in your pocket, and you have to give away 1 rupee each time a number different from '6' shows. You are *given* 5 rupees each time a '6' shows. Furthermore, you are allowed to *borrow* money in order to keep playing. What is the expected number of times you have to throw the dice before your pocket is empty?

In addition to the above rule, you are *given* 5 rupees each time a '6' shows. Furthermore, you are allowed to *borrow* money in order to keep playing.

Question: What is the expected amount of money in your pocket after 20 throws?

We define X as the random variable that counts the number of '6' faces thrown in 20 throws.

This is distributed according to the binomial distribution

$$P(X = r) = \binom{20}{r} \frac{5^{20-r}}{6^{20}}$$

Since you are allowed to keep a negative balance, the money in your pocket is

$$\begin{aligned}&(\text{starting money}) + (\text{money won from } X \text{ 6's}) \\ &\qquad - (\text{money lost from } (20 - X) \text{ non-sixes}) \\ &\qquad\qquad = 5 + 5X - (20 - X) = 6X - 15\,.\end{aligned}$$

(which could be negative). Its (mathematical) expectation is

$$E(6 \cdot X - 15) = 6E(X) - 15 = 6 \cdot 20 \cdot (1/6) - 15 = 5\,.$$

We note that this actually does not depend on the (fixed) number of throws. So, if, instead of 20 throws, there were 100 throws, the expected amount is still the same 5 rupees you started with!

This agrees with our intuition that getting 5 rupees for the '6' balances the losses of 1 rupee for each non-six. In other words, the expected win/loss from each throw is 0.

3. Difficult Case

Now, assume that you are *not* allowed to keep a negative balance. (In other words, when your pocket is empty you have to quit.)

There are two factors at play here, thus the expectation could increase or decrease.

Question: Will the expected amount in the previous question increase or decrease?

We define Y to be the random variable that counts the number of '6' throws during this *new* game. (It is always a good idea to use a different variable when what is being kept track of is different!)

There are two factors at play here:

- In order to have a positive amount of money in the pocket, the first '6' must occur *on or before* the 5th throw, the second '6' must occur *on or before* the 11th throw and the third '6' must occur *on or before* the 17th throw. It follows that (with X as in the previous game):

  $$P(Y = r) < P(X = r) \text{ for } r \geq 3$$

  This *decreases* the contribution to expectation from terms with $r \geq 3$.

- If $Y \leq 2$, then the pocket will contain 0 instead of $6Y - 15$ (which is less than 0). This *increases* the contribution to expectation from terms with $r \leq 2$.

Thus, the *expected amount could increase or decrease*. If we try to calculate it by trying to find the probability of various sequences of events, it quickly grows into an unmanageable mess! A different approach is required.

We will assume that the dice are thrown whether or not the player is in the game. (Maybe there is a dice throwing machine which does not stop!) All that the game rule says is that if the player has no money in the pocket then the throw of dice *no longer* leads to win or loss for the player! We can formulate this as follows.

Let W_i be the random variable indicating the amount of money won (negative indicating money lost) by the player on the i-th throw of the dice. The amount of money in the player's pocket *after* the i-th throw is $X_i = 5 + W_1 + \cdots + W_i$; as a result of the condition of the game, we have $X_i \geq 0$. The condition of the game can be stated as $W_{i+1} = 0$ given $X_i = 0$. Stated in terms of probability $P(W_{i+1} = 0|X_i = 0) = 1$. It follows that $E(W_{i+1}|X_i = 0) = 0$. On the other hand, if $X_i > 0$, then the $i + 1$-th dice throw *does* result in win or loss for the player. However, the expected win amount is also 0 in that case! In other words $E(W_{i+1}|X_i = r) = 0$ for $r > 0$ as well, or $E(W_{i+1}|X_i = r) = 0$ for any value of r. It follows easily that $E(X_{i+1}|X_i = r) = r$. Since we start with $X_0 = 5$, this that $E(X_{20}) = 5$.

In other words, even in this case, for a (fixed) number of throws, the expected amount in your pocket at the end of the game is 5 rupees. There is no change in your fortunes whether you are allowed to borrow money or not!

We note that if (as above) X_i denotes the amount of money in your pocket after the i-th throw, then we have

$$E(X_{i+1}|X_i) = E(X_i + W_{i+1}|X_i) = X_i + E(W_{i+1}|X_i) = X_i\,.$$

Such a sequence of random variables is quite useful in the study of games (as seen above) and is called a *martingale*.

Suggested Reading

[1] K B Athreya and M G Nadkarni, Mathematics of Risk Taking, *Resonance*, Vol.14, No.1, pp.66–79, 2009.

[2] Wikipedia article on "Martingales", (https://en.wikipedia.org/wiki/Martingale"_(probability_theory)

[3] K L Chung and F AitSahlia, *Elementary Probability Theory: With Stochastic Processes and an Introduction to Mathematical Finance*, Springer; 4th Edition, 2006.