Feynman Diagrams: A Toy Example*

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We examine the moments of the Gaussian integral and relate them to Feynman diagrams. We next introduce a quartic term and show how it leads to a seemingly paradoxical result. The article is addressed to the novice, but we believe that it may also serve as an opening lecture on the topic of Feynman diagrams.

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1. Introduction

This is the birth centenary of the eminent physicist and renowned teacher Richard P. Feynman (1918–1988). He successfully conveyed the joy of learning to scientists and students alike. It is in this spirit that we undertake the task of conveying to the uninitiated the meaning of Feynman diagrams [1]. This visual scheme in myriad forms has been used by countless theoretical physicists in particle and many-body physics [2]. While it is addressed to the novice, we believe that the initiated will find it an entertaining and useful pedagogical example for a beginning lecture on this topic.

2. Gaussian Integrals

The first basic result we need is the following:

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp[-x^2] = \sqrt{\pi} \, . \tag{1}$$

The way to derive this is to first compute I^2:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y \, \exp[-x^2 - y^2] \, . \tag{2}$$

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Now, we note that this is an integral over the plane $\mathbb{R}^2$, written in Cartesian coordinates on the plane. We can switch to polar coordinates (r, θ). Remember that $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$. The integrand does not depend on θ, so that integral is trivial. We are left with

$$I^2 = 2\pi \int_0^\infty r\mathrm{d}r \exp[-r^2] \,. \tag{3}$$

We can now do this integral by substitution $t = r^2$, and we get the result (1).

We can generalize this slightly to the formula

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp\left[-\frac{a}{2}x^2 + bx\right] = \sqrt{\frac{2\pi}{a}} \exp\left[\frac{b^2}{2a}\right] . \tag{4}$$

The Feynman diagrams are a powerful visual tool employed by countless physicists in particle and many-body theory.

The way to derive this is to complete the square in the exponent on the left hand side, and then shift and rescale the integration variable. For convergence, it is necessary that $a > 0$, whereas b can be any real number. In fact the above formula holds even if a and b are complex provided the real part of a is positive.

3. Moment Integrals of the Gaussian

One would also like to compute the integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^n \exp\left[-\frac{a}{2}x^2\right] . \tag{5}$$

We take n to be a positive integer. For odd n, the integrand is an odd function, and hence the integral is zero. For even n, one can use Feynman's favorite differentiate-under-the-integral-sign trick. On the left hand side of (4), taking derivatives with respect to b brings down factors of x. Thus to get to the integrand in (5), we can take n derivatives with respect to b in (4), and then set $b = 0$. The right hand side of (4) can also be expanded in b and contains terms of the form $(1/k!)(b^2/2a)^k$. The term we need is the one with $k = n/2$. This is the only term that will survive when we take n derivatives and set $b = 0$ (because this term is proportional to b^n). Thus, we see that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^n \exp\left[-\frac{a}{2}x^2\right] = \sqrt{\frac{2\pi}{a}} \frac{\mathrm{d}^n}{\mathrm{d}b^n} \exp\left[\frac{b^2}{2a}\right]\Big|_{b=0} \tag{6}$$

$$= \sqrt{\frac{2\pi}{a}} \frac{n!}{(n/2)!(2a)^{n/2}} \,. \tag{7}$$

One would like to think of $\exp[-ax^2/2]$ as a probability density. But as such, it is not properly normalized. The normalization constant is again obtained from (4) by setting $b = 0$. So we see that

$$\frac{\int_{-\infty}^{\infty} \mathrm{d}x\, x^n \exp\left[-\frac{a}{2}x^2\right]}{\int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left[-\frac{a}{2}x^2\right]} = \frac{n!}{(n/2)!(2a)^{n/2}} \,. \tag{8}$$

Moments like these generalize to so-called 'correlation functions' in probability theory, or even in quantum field theory.

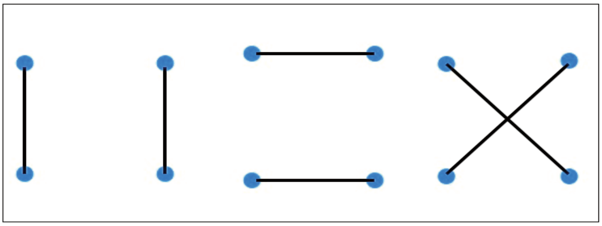
The evaluation of the moments of the Gaussian integral can be reduced to an enumeration of Feynman diagrams.

The answer in (8) can be given the following combinatoric interpretation. Let us place n dots on a piece of paper, where n is even. These dots are to be thought of as distinct, so one can imagine that each dot has a unique label. The combinatoric problem is: How many ways are there to make $n/2$ pairs out of these n dots? The answer is

$$\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}\cdots\binom{2}{2} \times \frac{1}{(n/2)!} = \frac{n!}{2^{n/2}(n/2)!} \,. \tag{9}$$

This is almost (8). We add in a factor of $1/a$ for each pair that we make by hand. Such factors are called 'propagators'. There is one propagator for each edge. To summarize: The combinatoric rule to compute normalized moments is to layout n dots (vertices of a graph), then draw edges making pairs of dots. There are many ways to pair up the dots. Each pairing gives us a 'diagram' or a 'graph'. To each diagram, we have to assign a weight. The weight is simply the product of propagators. Each edge gets a propagator, which in our example, is a factor of $1/a$. Such diagrams are also called 'Feynman diagrams' or 'Feynman graphs' (*Figure* 1). The full answer is given by adding the weights of all the graphs.

Figure 1. Feynman graphs.



For example, if $n = 4$ we have

$$\frac{n!}{2^{n/2}(n/2)!} = \frac{4!}{2^2 2!} = \frac{24}{4 \times 2} = 3\,. \tag{10}$$

The following are the three Feynman graphs:

Each graph has a weight $1/a^2$. The full answer is thus $3/a^2$.

4. Adding Interactions

Integrals such as (5) which only contain quadratic terms in the argument of the exponential are simple. So far, we could evaluate everything exactly. So one might wonder all this combinatoric nonsense is much ado about nothing. But now let us try to compute

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \exp[-ax^2/2 - \lambda x^4]\,. \tag{11}$$

Of course if $\lambda > 0$, the integrand is highly suppressed at large x, and thus the integral is perfectly well-defined. But now we do not have a simple trick as squaring the integral and doing it exactly.

However, we can use our exact results for the Gaussian integrals to make some progress when λ is small. When λ is small, we can simply Taylor expand in λ.

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \exp[-ax^2/2]\left(1 - \lambda x^4 + \frac{\lambda^2 x^8}{2} + \ldots\right). \tag{12}$$

Now, we can do the integral term by term using (8), since these are just moments of the Gaussian. So we see the utility of Feynman diagrams: If we want to compute the term in the answer proportional to λ^k, we simply compute the x^{4k} moment of the Gaussian. The 'perturbative' expansion of the integral is thus

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp[-ax^2/2 - \lambda x^4] = \sqrt{\frac{2\pi}{a}} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{(4k)!}{2^{2k}(2k)!} \frac{1}{a^{2k}} . \tag{13}$$

Note that we can make a quick dimensional check on our calculation. An examination of the left hand side informs us that the integral must have dimensions of x. The term λ/a^2 (or its powers) on the right hand side is dimensionless, and the term $\sqrt{a}$ in the denominator provides the requisite dimension of x.

This in brief is Feynman's program. In quantum field theory, one starts with an action principle. Making an analogy with the integral (11), we can say that free particles are described by quadratic terms like x^2. Terms with higher powers, like x^4 represent forces acting between particles. We can also consider other models of force, for example x^6, etc. The propagator is always the inverse of the quadratic term. When the forces are weak, one can resort to perturbation theory like we did for (11), and hope to get reliable answers. The curious reader is urged to read further [3]. In quantum-electrodynamics, the analog of λ is the fine structure constant $\alpha(= e^2/(4\pi\epsilon_0 c\hbar)) \approx 1/137$ (= 0.0073).

Free particles are described by the quadratic term, whereas the quartic term represents the interaction between particles.

Let us put this to a concrete test. Let us take $a = 1$ and $\lambda = 1/137$. The integral (evaluated numerically) is

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp[-x^2/2 - x^4/137] \approx 2.457479067\ldots . \tag{14}$$

Let us also look at the partial sums in (13) as displayed in *Table* 1.

What has gone wrong here? As we keep more and more terms, the answer seems to get better up to order λ^{15}, and then gets worse, and eventually looks like it is diverging. Puzzling. The situation is worse if λ is bigger. The more mathematically inclined reader is encouraged to apply the ratio and root test and convince herself that the radius of convergence of the series on the right hand side of (13) is zero. This is because of the exceptionally large $(4k)!$

Table 1. The partial sum of (13). The series seems to conver up to $k = 15$ and then seems to veer off wildly. Not all values for k are shown. The exact value up to nine decimal places is 2.457479067...

k	Partial Sum
0	2.50663
1	2.45174
2	2.45875
3	2.45706
14	2.45766
15	2.45717
16	2.45804
24	2.89611
25	1.21399
26	6.12672
32	5195.72
33	-19539.2
34	75805.4
35	-302849.0
36	1.24516×10^6

in the numerator, which overwhelms everything at large k. Such series which diverge but still give us useful approximations to some degree are called 'asymptotic' series [4].

The magnetic moment of the electron is predicted to a very high level of accuracy, and this is a triumph of the use of Feynman diagrams in QED.

We may ask: Where was the bug? Our derivation of (13) seemed perfectly reasonable. The bug is when we exchanged the sum and the integral after (12). In general, one is not allowed to interchange infinite sums and integrals. We may add that there are related terms to describe the above perturbation series – the saddle point approximation, the stationary phase approximation, and the method of steepest descent. One also encounters it in the WKB method for bound eigenvalues, but a discussion on this lies outside the main focus of this article.

In practice, in actual quantum field theory, even order λ^1 calculations are very hard. These are called 1-loop terms. Forget about order λ^{15}. And the first few orders provide an extremely good approximation to physical quantities. For example, consider the anomalous magnetic moment of the electron, which measures how strongly the spin of the electron couples to magnetic fields. As of 2016, the coefficients of the QED formula for the anomalous magnetic moment of the electron have been calculated up to order α^5:

$$a_e = 0.001\ 159\ 652\ 181\ 643(764) \qquad (15)$$

The QED prediction agrees with the experimentally measured value to more than 10 significant figures, making the magnetic moment of the electron the most accurately verified prediction in the history of physics. The current experimental value and uncertainty is:

$$a_e = 0.001\ 159\ 652\ 180\ 73(28) \qquad (16)$$

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Suggested Reading

[1] **Richard P Feynman, The Theory of Positrons *Physical Review*, Vol.76, pp.749–761, 1949.**

[2] **Richard D Mattuck, *A Guide to Feynman Diagrams in the Many-Body Problem*, New York: Dover Books, 1992.**

[3] **A Zee, *Quantum Field Theory in a Nutshell*, Second Edition, New Jersey: Princeton University Press, 2010.**

[4] **See Carl M Bender and Steven A Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, New York: McGraw-Hill, 1978.**

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