On Some of Jean Bourgain's Work*

B Sury and Kaushal Verma

The Belgian mathematician Jean Bourgain was born in Ostende in 1954. After a whirlwind career during which he solved many deep problems and transformed several areas of mathematics, he passed away in Bonheiden on 22nd December 2018 (the birth anniversary of Ramanujan). Bourgain was the modern-day equivalent of Leonhard Euler, making prolific contributions to a wide variety of problems in mathematics and physics. The *Mathematical Reviews* cite 511 publications under his name. Such a wide spectrum of work cannot be described in one article with any justice even if the authors were to possess expertise in a number of these areas. The breadth and depth of Bourgain's work can be fathomed from the following phrase used in a review by a renowned mathematician Ben Joseph Green; he said, "It is beyond the capability of the reviewer to give anything like a meaningful description of the argument here, save to repeat the authors' comments...." The review was of a paper that completely solved Vinogradov's mean value conjecture in analytic number theory. We choose a small assortment of the topics to which he contributed so deeply and, describe some technical details in rough terms. In the end, we make a brief mention of his results in a wide variety of areas (see [3–5], [7–14] for technical details). The topics we dwell on in this write-up include the so-called Kakeya problem and some striking applications to number theory.

Sury likes interacting with talented high school students and writing for undergraduate students. At present, he is the National Co-ordinator for the Mathematics Olympiad Programme in India.

Kaushal Verma is with the Department of Mathematics, Indian Institute of Science, Bengaluru. His research interest is in complex analysis.

1 Zaremba's Conjecture, Apollonius Circles and Thin Groups

Bourgain along with other collaborators like Gamburd, Kontorovich

Keywords
Kakeya problem, number theory, Apollonian circle, Zaremba's conjecture, Pythagorean triples.

*DOI: https://doi.org/10.1007/s12045-019-0808-2

and Sarnak has made spectacular discoveries in diverse arithmetic problems. Among the most striking arithmetic problems to which these discussions apply are:

(i) Integral Apollonian circle packings,
(ii) Zaremba's conjecture on continued fractions and,
(iii) The determination of primitive Pythagorean triples whose hypotenuse (largest integer) is prime.

Several basic number-theoretic problems reduce to equivalent assertions about groups of matrices. More often than not, they lead to properties (like local-global principles) of orbits of a Zariski dense subgroup – often of infinite index – in an arithmetic group.

Several basic number-theoretic problems reduce to equivalent assertions about groups of matrices. More often than not, they lead to properties (like local-global principles) of orbits of a Zariski dense subgroup – often of infinite index – in an arithmetic group. Here, "arithmetic subgroups" are typically groups of matrices with entries in the ring of integers. The notion of a subgroup being Zariski dense in a group roughly means that any polynomial function on the big group that is zero on the elements in the subgroup must be identically the zero function. The phrase'*thin subgroup*' is used for a Zariski dense subgroup of infinite index in an arithmetic group. For instance, consider the subgroup of $SL(2,\mathbb{Z})$, the 2×2 integer matrices of determinant 1, generated by the two matrices $x = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ - this is easily shown to be a free group. It is also a 'thin' subgroup. Thus, thin subgroups are small in the sense of being of infinite index but nevertheless large in the sense of being Zariski dense. The thin groups have a striking property called "Strong approximation" – roughly, the technical general version of Chinese remainder theorem. This property is key for tackling many difficult classical number theoretic problems.

Let us first describe the Apollonian circle packing problem (i) above. Given three circles mutually tangent at 3 distinct points, there are exactly two other circles each of which is tangent to the original three circles. Iterating this construction of Apollonius leads in the limit to what is known as the Apollonian gasket. At each step, the circles shrink in radius and one natural question is to determine (for a given gasket; that is, starting with a given configuration of four such circles) how many circles asymptotically arise with curvatures (reciprocals of radii) bounded by a parameter as $T \to \infty$. Further, if the four starting circles have integer curvatures, one may ask how many distinct integers occur (as curvatures of circles in the gasket) up to a parameter N as $N \to \infty$.

To reformulate these questions in the language of thin orbits, we recall Apollonius's observation that the starting curvatures $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ satisfy the so-called Descartes quadratic form Q; that is,

$$2\sum_{i=1}^{4} \kappa_i^2 - (\sum_{i=1}^{4} \kappa_i)^2 = 0.$$

Over the reals, this quadratic form is of type (3, 1). The convention is to give signs to the curvatures by giving the opposite sign to the biggest (bounding) circle among the four starting ones. If the 3-tuple $(\kappa_1, \kappa_2, \kappa_3)$ is given, then there are two roots of the above equation for κ_4; if the other root is κ_4', then

$$\kappa_4 + \kappa_4' = 2\sum_{i=1}^{3} \kappa_i.$$

In other words, given one starting 4-tuple $v_0 := (\kappa_1, \kappa_2, \kappa_3, \kappa_4)^t$, we have four other 4-tuples $S_1 v_0, S_2 v_0, S_3 v_0, S_4 v_0$ where

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $S_4 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

The four matrices above belong to the group O_Q and generate the so-called Apollonian group Γ. The orbit Γv_0 comprises of all 4-tuples of curvatures of 4 mutually tangential circles which occur in the gasket which has the starting 4-tuple v_0. If the four initial curvatures in v_0 are integers, then all the elements of its orbit have integer entries because $\Gamma \leq O_Q(\mathbf{Z})$. The Apollonian group is a 'thin' subgroup of the arithmetic group $O_Q(\mathbb{Z})$. The study of such 'thin' orbits Γv_0 leads to information on the two questions posed above.

Besides that, it also (essentially) proves a conjecture due to Graham, Lagarias, Mallows, Wilks & Yan. Noticing that certain local congruence conditions are necessarily forced on the curvatures which occur, they conjectured that every sufficiently large "*admissible*" (that is, satisfying the local congruence conditions) number occurs as a curvature in the gasket. Bourgain and Kontorovich have shown that this conjecture holds for almost (in the sense of density) every admissible number.

The Zaremba conjecture arose from a study of good lattice points for quasi-Monte Carlo methods in numerical integration. It asserts that there is an absolute constant $A > 1$ such that for every natural number d, there is some b relatively prime to d with continued fraction expansion $\frac{b}{d} = [a_1, \cdots, a_k]$ where $a_i \leq A$ for all k.

Let us now describe another striking problem – the Zaremba conjecture referred to as (ii) above. The conjecture arose from a study of good lattice points for quasi-Monte Carlo methods in numerical integration. It asserts that there is an absolute constant $A > 1$ such that for every natural number d, there is some b relatively prime to d with continued fraction expansion $\frac{b}{d} = [a_1, \cdots, a_k]$ where $a_i \leq A$ for all k. This is essentially equivalent to asserting that every natural number occurs as the denominator of a reduced fraction whose partial quotients (the a_i's above) are bounded by an absolute constant A. Zaremba suggested the value $A = 5$ as a possibility. Bourgain et al. proved that almost every (in the sense of density) natural number is the denominator of a reduced fraction whose partial quotients are bounded by 50. If C_A denotes the set of limit points of the set of rationals $\frac{b}{d} = [a_1, \cdots, a_k]$ with $a_i \leq A$ for all k, then the analysis depends on the Hausdorff dimension of C_A. The problem is once again of looking at an orbit of a thin group. In this case, one has the orbit of $(0, 1)^t$ under a *semigroup*; this is the semigroup generated by the matrices $\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$ where $a \in A$. This is not surprising because the equality $\frac{b}{d} = [a_1, \cdots, a_k]$ is usually expressed as

$$\begin{pmatrix} * & b \\ * & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix}.$$

Finally, we describe the third problem on prime solutions of Diophantine equations alluded to as (iii) above. If one considers the primitive Pythagorean triples $u^2 - v^2, 2uv, u^2 + v^2$ with u even and v odd, the determination of those admissible primes which occur as $u^2 + v^2$ amounts to the analysis of the orbit $\Gamma(2, 1)^t$ where the thin group Γ is generated by $\pm\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ and $\pm\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$.

For instance, one may ask if there are infinitely many admissible primes. *This is still unknown.* On the other hand, one could look at other functions on the orbit (in the above problem, we have mentioned the hypotenuse function). If one looks at a linear function f of u, v, Bourgain et al. proved that for any relatively prime u, v, there exists $\delta_0 < 1$ (for instance, 0.99995 suffices) such that if the Hausdorff dimension of the limit set of Γ is larger than δ_0, then almost every admissible number is represented in $f(\Gamma(2.1)^t)$.

The reformulations of the above three problems were unified and studied by Bourgain and others. Before the advent of the powerful affine linear sieve methods into these "nonabelian" situations, nobody could even start tackling such problems! The sieve methods (and something called the sum product formula) allows Bourgain and his collaborators to answer questions such as the following one:

Given an N, how many (in terms of N) matrices

$$x^{a_1} y^{b_1} \cdots x^{a_r} y^{b_r}$$

with $\sum_i(|a_i|+|b_i|) \leq N$ *have all entries primes (or almost primes)?* In this subject, it is convenient to talk about 'almost primes' meaning that for a fixed d, an almost d-prime is a number that is a product of at most d (not necessarily distinct) primes.

The deep and fundamental question of infinitude of real quadratic fields of class number due to Gauss is another area where Bourgain has substantial contributions.

2 The Gauss Problem on Real Quadratic Fields

The deep and fundamental question of infinitude of real quadratic fields of class number due to Gauss is another area where Bourgain has substantial contributions. The method is again a shift of the object of study to a geometric one this time. Let us explain this. Let $D > 0$ be the discriminant of a real quadratic field $K_D :=$

$\mathbf{Q}(\sqrt{D})$; associated to this is a finite group C_D called its class group, which measures how far the numbers $a + b\sqrt{D}$ are from having unique factorization where a, b are integers. To each element of C_D, one naturally associates a closed geodesic in the unit tangent bundle $X = T^1(PSL(2,\mathbf{Z})\backslash\mathcal{H}) \cong PSL(2,\mathbf{Z})\backslash PSL(2,\mathbf{R})$ of the modular surface. The closed geodesics arising in this manner are called fundamental. Bourgain and Kontorovich proved that there exists a compact subset $Y \subset X$ which contains infinitely many fundamental geodesics. This can be expressed as saying that there are infinitely many fundamental geodesics which are *low-lying*; that is, do not travel high on the cusp – when we interpret compactness in X. An old famous theorem due to W.Duke establishes the equidistribution of closed geodesics on the modular surface. This implies the following.

There are infinitely many fundamental geodesics which are *low-lying*; that is, do not travel high on the cusp – when we interpret compactness in X.

Let μ_X denote the invariant probability measure on X. For each class γ in the class group C_D, consider the corresponding fundamental geodesic and the associated arc length probability measure μ_γ. Duke's work implies that

$$\frac{1}{h_D}\sum_{\gamma\in C_D}\mu_\gamma \overset{weak*}{\longrightarrow} \mu_X$$

as $D \to \infty$.

That is, μ_γ's are equidistributed on average over C_D; the abovementioned result addresses the question of equidistribution at the level of individual closed geodesics. Note that for real quadratic fields of class number 1, the above sum reduces to a single term. For each $\epsilon > 0$, there is a compact region $Y(\epsilon) \subset X$ containing infinitely many fundamental geodesics.

To state the result precisely, let $\epsilon > 0$. Then, there is a compact region $Y(\epsilon) \subset X$, and a set $D(\epsilon)$ of fundamental positive discriminants such that the following two properties hold:
(i) for each $D \in D(\epsilon)$, $\#\{\gamma \in C_D : \gamma \in Y(\epsilon)\} > C_D^{1-\epsilon}$;
(ii) $\#(D(\epsilon) \cap [1, T]) > T^{\frac{1}{2}-\epsilon}$ as $T \to \infty$.

As in the solution to Zaremba's conjecture, the key is to convert the problems to those on continued fractions and then to thin orbits and to affine sieves. Fundamental geodesics lead to associated continued fractions by means of the cutting sequence of the geodesic flow. The condition that a fundamental geodesic is low-lying (that is, lies in $Im(z) \leq C$ for some $C > 0$) gets translated to the condition that all the partial quotient of the corresponding periodic continued fraction are bounded by a constant $A = A(C)$.

3 Recent work on Traces

In a paper published in 2018, Bourgain and Kontorovich carried these ideas even further and obtained the following results.

Their results are formulated in terms of a local global theorem for traces of a sub-semigroup of $SL_2(\mathbf{Z})$. For an alphabet $\mathcal{A}$ (a *finite* subset of $\mathbb{N}$), consider all the periodic continued fractions whose partial quotients belong to the alphabet set $\mathcal{A}$. When the number of alphabets is > 1, this set of real numbers is known to be a Cantor set of Hausdorff dimension $\delta_{\mathcal{A}} \in (0, 1)$.

As mentioned above, one converts problems on the periodic continued fraction $[\overline{a_0, a_1, \cdots, a_l}]$ of a quadratic irrational to problems on thin orbits of semigroups of matrices. In fact, since the matrix

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_l & 1 \\ 1 & 0 \end{pmatrix}$$

fixes the quadratic irrational $[\overline{a_0, a_1, \cdots, a_l}]$, one looks at the *semi-group* $\Gamma_{\mathcal{A}}$ of $GL(2, \mathbf{Z})$ generated by the matrices $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ for $a \in \mathcal{A}$. By intersecting with $SL_2(\mathbb{Z})$, we obtain a semigroup that is thin. Bourgain and Kontorovich studied the multiplicities of the set $T_{\mathcal{A}}$ of positive integers that are traces of the matrices in the above semigroup; in other words, they consider for each positive integer t, the set $M_{\mathcal{A}}(t) = |\{M \in \Gamma_{\mathcal{A}} : trace(M) = t\}|$.

The following general conjecture is stated by them:

Local-Global Conjecture for Traces. For an arbitrary alphabet $\mathcal{A}$ whose Hausdorff dimension is $> 1/2$, the set $T\mathcal{A}$ contains every sufficiently large integer t. Moreover, if $t \in T_{\mathcal{A}} \bmod q$ for each $q \geq 1$ (that is, there are no local obstructions), and $t \in [N, 2N)$, then the multiplicity $M_{\mathcal{A}}(t) > N^{2\delta_{\mathcal{A}}-1-o(1)}$.

The principal result obtained in this recent paper is the following progress towards the conjecture which establishes levels of distribution for $T_{\mathcal{A}}$ beyond those available from expansion alone. In particular, they recover a Ramanujan-type exponent unconditionally! They prove:

For any small $\eta > 0$, there is an effectively computable $\delta_0(\eta) < 1$ so that, if $\mathcal{A}$ is an alphabet whose Hausdorff dimension exceeds $\delta_0(\eta)$, then the set $T_{\mathcal{A}}$ has exponent of distribution $\frac{1}{3} - \eta$.
(We don't recall here the precise definition of the exponent of distribution).

Using standard sieve theory, they deduce the striking result:
There is an effective constant $\delta_0 < 1$ such that for each alphabet whose Hausdorff dimension exceeds, the set of traces contains infinitely many 4-almost primes.

Further, considering instead of the trace set, the set $D_{\mathcal{A}} = \{$square-free part of $t^2 - 4 : t \in T_{\mathcal{A}}\}$ of discriminants, combining with the work on the Zaremba conjecture, they can deduce:

For the alphabet $\mathcal{A} = \{1, 2, \cdots, 50\}$, the set $D_{\mathcal{A}}$ contains infinitely many 2-almost primes.

4 Spectral Gap Theorems

Bourgain along with Gamburd proved what is known as a spectral gap theorem for $SU(d)$. In principle, such a result has applications as diverse as to the theory of quasicrystals, the theory of quantum computation and to the Banach–Ruziewicz problem.

Bourgain along with Gamburd proved what is known as a spectral gap theorem for $SU(d)$. In principle, such a result has applications as diverse as to the theory of quasicrystals, the theory of quantum computation and to the Banach–Ruziewicz problem.

For a finitely generated subgroup of $SU(d)$ ($d \geq 2$) - generated by $\{g_1, \cdots, g_k\}$, say - one has the averaging operator (also called Hecke operator) on the Hilbert space $L^2SU(d)$:

$$f \mapsto z_{g_1,\cdots,g_k} f : x \mapsto \sum_{i=1}^{k} (f(g_i x) + f(g_i^{-1} x))$$

Evidently, any constant function is an eigenfunction with eigenvalue $2k$. One says that this Hecke operator has a spectral gap if the supremum of the eigenvalues on the orthogonal complement of the constants is strictly smaller than $2k$. The main result in this direction is:

Let $g_1, \cdots, g_k \in SU(d)$ have algebraic entries and generate a dense subgroup (equivalently, the group generated is Zariski dense in SL_d - meaning that any polynomial function on SL_d which is zero on this subgroup must be identically zero). Then, the associated Hecke operator has a spectral gap.

5 The Kakeya Conjecture

Let us now touch upon some of Bourgain's significant contributions to Kakeya-type problems. To set the context for this, recall the problem proposed by the Japanese mathematician S Kakeya in 1917: *find in the class of planar figures, the one with smallest area, that allows a unit line segment to be turned around by 180 degrees with the condition that the segment always stays within the given figure while being rotated.* Here, the area of a figure refers to its Lebesgue measure. In 1927, Besicovitch gave a surprising answer by showing that it is possible to have such sets with arbitrarily small area – the main idea was to construct certain planar sets that contain segments of a fixed length and a range of slopes (that would allow for the rotation to take place) in a way that maintains control over their area. This motivates the following definition – a subset of $\mathbb{R}^n$ is said to be a Besicovitch set if it contains a unit line segment in all possible directions. For each $n \geq 2$, the existence of Besicovitch sets with arbitrarily small (and in fact even zero) Lebesgue measure is known. Indeed, the original construction of Besicovitch gives such a set when $n = 2$, and by taking the product of this set with $\mathbb{R}^{n-2}$ produces a Besicovitch set in $\mathbb{R}^n$. While all this is perfectly fine and exciting, a detailed study of such sets requires a deeper understanding of their metric dimensions.

Find in the class of planar figures, the one with smallest area, that allows a unit line segment to be turned around by 180 degrees with the condition that the segment always stays within the given figure while being rotated.

– Kakeya Problem, 1917.

There are two ways to go about this; one can consider either their Hausdorff dimension or their (upper and lower) Minkowski dimension. We will consider only the latter for the sake of simplicity. The main question about Besicovitch sets is the so-called Kakeya conjecture which can be stated as follows: *Every Besicovitch set in $\mathbb{R}^n$ has Minkowski dimension n.*

In the early '90's, Bourgain introduced the so-called 'bush argument' to show that the Minkowski dimension of a Besicovitch set $E \subset \mathbb{R}^n$ is at least $(n+1)/2$. The rough idea is to consider a maximal δ-separated ($\delta > 0$) set of unit directions $v \in S^{n-1}$ and the corresponding unit line segments $T_v \subset E$. Let T_v^δ be a δ-neighbourhood of T_v and E_δ the union of the T_v^δ's. Estimating the volume of E_δ in two different ways shows that there is at least one point p that belongs to very many of these tubes – 'many' being quantifiable in terms of δ and the supposed Minkowski dimension of E. Since two lines can intersect at most in a point, the essential observation now is that these intersecting tubes have bounded overlap away from p. This leads to the lower bound of $(n+1)/2$ for the Minkowski dimension of E. This line of reasoning clearly resonates with existing results in combinatorics that deal with the possible number of intersections between a given configuration of lines and points, and while this view point did lead to some progress towards resolving the Kakeya conjecture, it was clearly not enough to affirmatively answer it.

Bourgain introduced new ideas from additive number theory to tackle the Kakeya conjecture.

In 1998, Bourgain introduced new ideas from additive number theory to tackle the Kakeya conjecture. Roughly speaking, his 'three-slice' method consists of intersecting the maximal δ-separated set (as above) with three distinct equidistant planes, say A, B, C which are respectively described by the equations $x_n = 0, 1, 1/2$ after re-scaling. Essentially each of the line segments T_v intersect A, B, C at unique points and hence the collection of T_v's can be identified with a subset, say G of $A \times B$. The set of sums $\{a + b : (a, b) \in G\}$ corresponds to the mid-points of these segments and hence should be contained in a neighbourhood of C. The cardinalities of A, B and hence C (being the mid-point set) can be estimated in terms of the Minkowski dimension of E. The set of differences $\{a-b : (a, b) \in G\}$ is however very big since this corresponds to the slopes of the lines T_v. The end-game then is to prove rather refined inequalities (using ideas from the work of W.

T. Gowers) that relate the cardinalities of sum and difference sets and use them to get bounds for the Minkowski dimension of E. This has led to much improved lower bounds for the Minkowski dimension of E specially in higher dimensions.

The excellent articles by Łaba [1] and Tao [2] contain many more details and useful references related to this theme.

6 Brief Mention of Other work

Some of the main contributions of Jean Bourgain's prolific work are described briefly as:

(i) Proving the first restriction estimates beyond Stein-Tomas and related contributions towards the Kakeya conjecture;
(ii) Proof of the boundedness of the circular maximal function in two dimensions;
(iii) Proof of dimension free estimates for maximal functions associated to convex bodies;
(iv) Proof of the pointwise ergodic theorem for arithmetic sets;
(v) Development of the global well-posedeness and uniqueness theory for the NLS with periodic initial data;
(vi) Proof that harmonic measure on a domain does not have full Hausdorff dimension;
(vii) Proof with Milman of Mahler's reverse-Santalo conjecture in convex geometry.

Bourgain's 1994 Fields Medal citation reads:

"*Bourgain's work touches on several central topics of mathematical analysis: the geometry of Banach spaces, convexity in high dimensions, harmonic analysis, ergodic theory, and finally, nonlinear partial differential equations from mathematical physics.*"

While the Fields Medal citation mainly focuses on Bourgain's contributions to harmonic analysis from the late 1980s and early 1990s, his contributions to Banach space theory in the early 1980's is equally significant and impactful.

While the Fields Medal citation mainly focuses on Bourgain's contributions to harmonic analysis from the late 1980s and early 1990s, his contributions to Banach space theory in the early 1980s is equally significant and impactful.

Suggested Reading

[1] Izabella Łaba, From Harmonic Analysis to Arithmetic Combinatorics, *Bull. Amer. Math. Soc.*, (N.S.) Vol.45, No.1, pp.77–115, 2008.

[2] Terence Tao, From Rotating Needles to Stability of Waves: Emerging Connections Between Combinatorics, Analysis, and PDE, *Notices Amer. Math. Soc*. Vol.48, No.3, pp.294–303, 2001.

[3] Jean Bourgain and Alex Kontorovich, Beyond Expansion IV: Traces of Thin Semigroups, *Discrete Anal*., Paper No.6, 27 pp. 2018.

[4] Jean Bourgain and Alex Kontorovich, Beyond Expansion II: Low-lying Fundamental Geodesics, *J. Eur. Math*., Soc. (JEMS) Vol.19, No.5, pp.1331–1359, 2017.

[5] Jean Bourgain, A Quantitative Oppenheim Theorem for Generic Diagonal Quadratic Forms, *Israel J. Math*., Vol.215, No.1, pp.503–512, 2016.

[6] Jean Bourgain, Ciprian Demeter, and Larry Guth, Proof of the Main Conjecture in Vinogradov's Mean Value Theorem for Degrees Higher Than Three, *Ann. of Math*., Vol.2, 184, No.2, pp.633–682, 2016.

[7] Jean Bourgain and Alex Kontorovich, The Affine Sieve Beyond Expansion I: Thin Hypotenuses, *Int. Math. Res. Not. IMRN*, No.19, pp.9175–9205, 2015.

[8] Jean Bourgain, Alex Kontorovich, On Zaremba's Conjecture, *Ann. of Math*., Vol.2, 180, No.1, pp.137–196, 2014.

[9] Alex Kontorovich, From Apollonius to Zaremba: Local-global Phenomena in Thin Orbits, *Bull. Amer. Math. Soc*. (N.S.), Vol.50, No.2, pp.187–228, 2013.

[10] Alex Kontorovich Hee Oh, Almost Prime Pythagorean Triples in Thin Orbits, *J. Reine Angew Math*. Vol.667, pp.89–131, 2012.

[11] J Bourgain, Integral Apollonian Circle Packings and Prime Curvatures, *J. Anal. Math*., Vol.118, No.1, pp.221–249, 2012.

[12] J Bourgain and A Gamburd, A Spectral Gap Theorem in SU(d)., *J. Eur. Math. Soc. (JEMS)*, Vol.14, No.5, pp.1455–1511, 2012.

[13] Jean Bourgain and Peter P Varju, Expansion in SLd(Z/qZ),q Arbitrary, *Invent. Math*., Vol.188, No.1, pp.151–173, 2012.

[14] Jean Bourgain, Alex Gamburd and Peter Sarnak, Generalization of Selberg's 316 Theorem and Affine Sieve, *Acta Math*., Vol.207, No.2, pp.255–290, 2011.

[15] Jean Bourgain, and Elena Fuchs, A Proof of the Positive Density Conjecture for Integer Apollonian Circle Packings, *J. Amer. Math. Soc*, Vol.24, No.4, pp.945–967, 2011.

Address for Correspondence
B Sury
Stat-Math Unit
Indian Statistical Institute
8th Mile Mysore Road
Bengaluru 560 059 India.
Email: surybang@gmail.com

Kaushal Verma
Department of Mathematics
Indian Institute of Science
Bengaluru 560 012, India.
Email: kverma@iisc.ac.in