(K)not So Abstract![*]

Knots, Links, and Three-dimensional Manifolds

Swatee Naik

This article is intended to give the reader a flavor of three-dimensional topology in an informal setting.

Swatee Naik is a Professor of Mathematics at the University of Nevada, Reno, and a Program Officer at the National Science Foundation, USA. Her research is in knot theory and low dimensional topology.

1. What is Topology and Why Study it?

Topology is a branch of mathematics, sometimes referred to as 'rubber-sheet geometry' based on the understanding that objects that can be made to look exactly like each other by stretching, bending, or repositioning without cutting, tearing, or poking holes, are 'topologically equivalent'. Truth be told, while establishing topological equivalence, sometimes even cutting or tearing is allowed as long as the tear is properly repaired, as we shall soon see.

Topology is not only relevant in several areas of mathematics, but it also has applications in physical sciences. Biologists call on topological methods to examine the effects of certain enzymes on DNA and to study structures of neural networks; in modern chemistry, altering the chemical and physical properties of compounds may be achieved through the synthesis of topologically different molecules; configuration spaces used in robotics incorporate topology; topological techniques for dimension reduction and robustness against noise have proved extremely useful in analysis of large data sets that we nowadays routinely encounter; quantum field theories and gauge theories that originated in physics

Keywords
Knots, links, topology, three dimensional manifolds, homeomorphism.

[*]DOI: https://doi.org/10.1007/s12045-019-0785-5

This material is based upon work supported by and while serving at the National Science Foundation. Any opinion, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

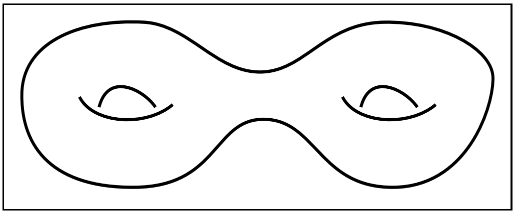

Figure 1. A double doughnut is a genus 2 handlebody.

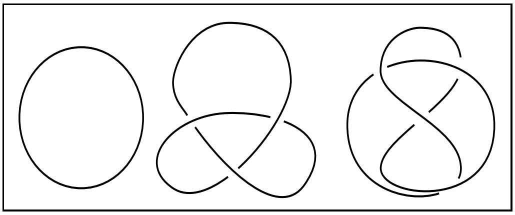

Figure 2. Three knots: or are they?

are closely connected to three and four-dimensional manifolds (special type of topological spaces to be defined later), and their studies have immensely benefitted both mathematics and physics.

In essence, what topology captures is not the exact shape or size of an object but deeper structural properties, one such being the number of 'handles'.

In the Science Smiles Section of March 2018 issue of *Resonance* [1], it was mentioned that a coffee cup and a doughnut are the same for a topologist. In essence, what topology captures is not the exact shape or size of an object but deeper structural properties, one such being the number of 'handles'. Both a coffee cup and a traditional doughnut or a *medhuwada* have one handle each, but the double doughnut seen in *Figure* 1 has two, making it topologically inequivalent to the coffee cup.

A topological equivalence between objects or (topological) spaces is a continuous, one-to-one, onto function from one to the other with a continuous inverse. Such a function is called a 'homeomorphism', and equivalent spaces are said to be homeomorphic. We need mathematical structure to make sense of continuity, but as our examples are subsets of Euclidean spaces, continuous functions can be intuitively understood as ones that take 'nearby points to nearby points'.

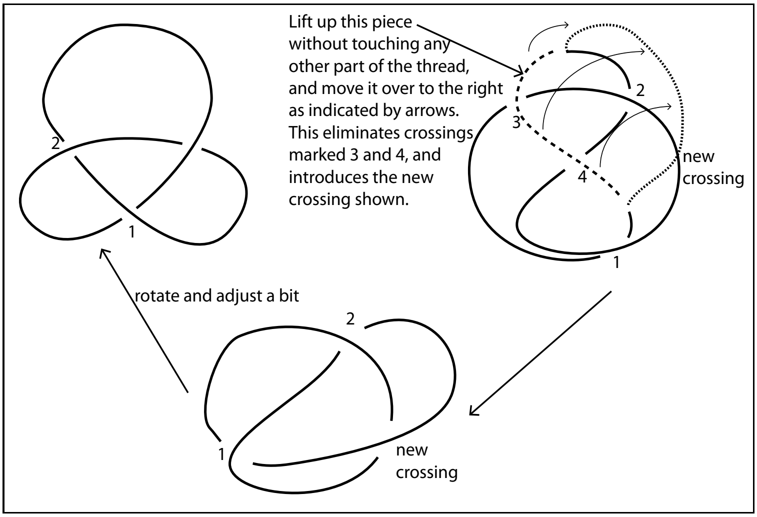

Figure 3. Diagram moves for the right-handed trefoil.

2. What Knot?

The curves shown in *Figure* 2 are all homeomorphic to each other. The homeomorphism can be seen in steps: First, snip the thread in any one place, then untangle the knot if there is one, and finally glue the snipped ends back together to repair the cut. In this process, points on the thread that were nearby have remained so, preserving continuity. Clearly, this can be done to any knot. But as we shall see, there is more to the story. The knot on the right in *Figure* 2 can be transformed into the one in the middle without any snipping as shown in *Figure* 3. Conclusion? These two knots are equivalent to each other through a relationship that is stronger than a homeomorphism between knots. This is what 'knot theory', a subbranch of topology is built on.

Knot theory studies knotted one-dimensional circles in the three-dimensional space. Two knots K_1 and K_2 are considered equivalent, or isotopic if there is a homeomorphism from the entire space to itself that maps one knot to the other. A cut causes damage to the surrounding space as well as to the knot, and the damage to the space is not repaired when we make a local repair on the knot. If a knotted circle is transformed into another without cutting, that sequence of moves corresponds to a homeomorphism

Knot theory studies knotted one-dimensional circles in the three-dimensional space. Two knots K_1 and K_2 are considered equivalent, or isotopic if there is a homeomorphism from the entire space to itself that maps one knot to the other.

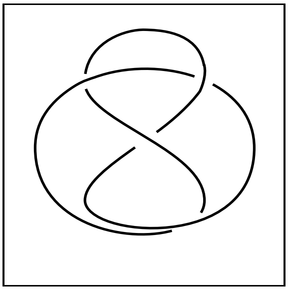

Figure 4. The figure 8 knot.

of the surrounding space to itself.

The knot in *Figure* 4 is not the same as (i.e. not isotopic to) the right handed trefoil, depicted on the right in *Figure* 2. Can you see the difference?

Knot diagrams are projections in the plane with under-over crossing information that tells us about positioning in space.

The pictures we have drawn are called the 'knot diagrams'. These are two-dimensional projections with under-over crossing information that tells us about positioning in space. Two diagrams represent isotopic knots, if and only if, one can be transformed into the other as in *Figure* 3 without cutting.

Scientists originally became interested in knots because of theories about the physical universe, which have since become outdated. Mathematical interest in knot theory continued through their applications in algebraic geometry and topological studies of 3- and 4-dimensional manifolds, eventually returning to a myriad of new real-world applications. Mathematicians have proved that by removing a thickened knot (a knotted solid torus) from a three-dimensional space and filling in the hole differently, we can obtain different kinds of three-dimensional manifolds. We will return to this in Section 5.

3. Managing the Surroundings

So far, we have vaguely referred to the space surrounding a knot as three-dimensional, giving the impression that we are working within the Euclidean space $\mathbb{R}^3$ with x, y, z axes. Euclidean spaces extend to infinity in all directions. It is easier to work within more manageable spaces called the 'compact manifolds' that we

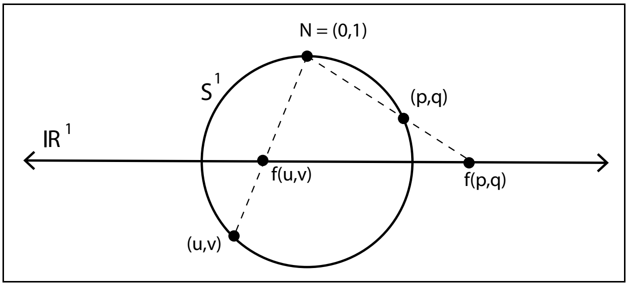

Figure 5. A circle minus a point is homeomorphic to a line .

describe below.

An n-dimensional manifold is a subset of some Euclidean space $\mathbb{R}^m$ (typically $m > n$) with the property that near any point, it 'looks like' an n-dimensional Euclidean space. For subsets of $\mathbb{R}^m$, compact means 'closed' and 'bounded', where closedness is characterized by every convergent sequence of points in the set having a limit inside the set itself, and boundedness indicates that the entire set lies within a finite distance from the origin.

Closed and bounded subsets of a Euclidean space are compact.

For example, the circle $S^1 = \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\right\}$ is a compact, 1-dimensional manifold: around any point of S^1 one can draw an 'open interval', just like on the number line!

Do you see that the standard sphere S^2 is a 2-dimensional manifold?

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\right\}.$$

Now, consider $f\colon S^1 - \{(0, 1)\} \to \mathbb{R}$ given by $f(x, y) = (\frac{x}{1-y}, 0)$. What this function does can be seen gemotertically in *Figure* 5.

It maps a point (a, b) on the circle (minus its north pole N) to the point where the line joining N with (a, b) intersects the x-axis. It is easy to show that this defines a homeomorphism from the circle with a point removed to the real line. Taking it backwards, the real line is homeomorphic to a subset of the circle that misses only one point on the circle. We can envision this phenomenon as the two sides of the real line bending up towards $(0, 1)$. The 'point at infinity' that both 'ends' of $\mathbb{R}$ approach is the point $N = (0, 1)$ in $\mathbb{R}^2$.

You may have encountered *stereographic projection* that maps a point, say $(p, q, r) \neq (0, 0, 1)$ on S^2 to the point in the xy-plane that the line joining $(0, 0, 1)$ and (p, q, r) intersects, and provides a homeomorphism between S^2 minus a point and the Euclidean plane $\mathbb{R}^2$. Imagine the infinite plane bending upwards with its edges trying to reach the point at infinity, which happens to be conveniently located at $(0, 0, 1)$ in $\mathbb{R}^3$.

The general concept at play here is that an n-dimensional sphere is a *one-point compactification* of the n-dimensional Euclidean space. In particular, the three-dimensional sphere

$$S^3 = \left\{ (x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1 \right\},$$

is a compact 3-dimensional manifold, and S^3 with a point removed is homeomorphic to $\mathbb{R}^3$. We interchangeably view knots as residing within $\mathbb{R}^3$ or S^3 and continue to use knot diagrams in the plane. If two knots are isotopic in S^3, they are so in $\mathbb{R}^3$, and vice versa.

At times it may be convenient to rely on formulas from algebra and geometry, but shapes are not fixed in topology.

This is a good place to reiterate that although at times it may be convenient to rely on formulas from algebra and geometry, shapes are not fixed in topology. Objects can be moved, stretched, and deformed by means of homeomorphisms. The radius of a topological circle can be as large or as small as the situation calls for, and S^1 could even be an ellipse or any other simple, closed curve. For our discussion, let's agree to reserve the term 'circle' for *unknotted*, simple, closed curves.

4. Product Spaces and an Alternative Description of S^3

Any point in the Euclidean plane $\mathbb{R}^2$ can be located by its x and y coordinates. Moreover, any function whose codomain is $\mathbb{R}^2$ is continuous if and only if its *projections* onto the axes are continuous. This is an example of a *product space* that can be described in terms of its *coordinate spaces*. In case of $\mathbb{R}^2$ both coordinate spaces are $\mathbb{R}$, represented by the x- and the y-axis, respectively. Likewise, the 4-dimensional Euclidean space $\mathbb{R}^4$ can be viewed as a product space $\mathbb{R}^2 \times \mathbb{R}^2$. If the coordinate spaces X and Y are

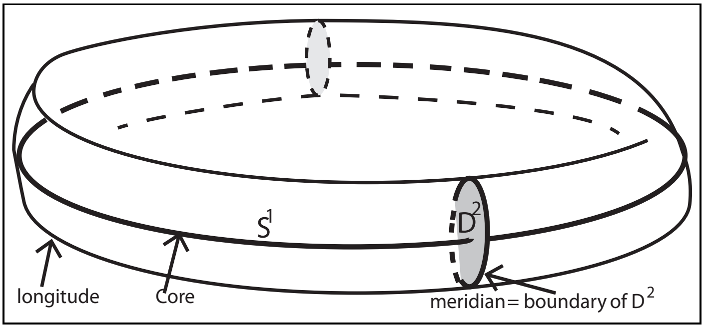

Figure 6. A solid torus.

distinct spaces, the product space $X \times Y$ happens to be homeomorphic to $Y \times X$.

Can you see that a solid torus is spanned by a 2-dimensional disk D^2 rotated around a circle S^1? As we see in *Figure* 6, there is a circle corresponding to each point of the disk and the solid torus is a union of these circles.

Can the core of a solid torus be "longer" than a parallel circle on the boundary?

We can write 'coordinates' for a point on the solid torus by specifying a point of S^1 and one of D^2. In this sense, a solid torus is (homeomorphic to) a product space $S^1 \times D^2 \subset \mathbb{R}^2 \times \mathbb{R}^2 \subset \mathbb{R}^4$. The boundary of this solid torus is a product of two circles $S^1 \times S^1$. One of these bounds a disk within the solid torus; this circle is called a 'meridian'. The other S^1 that intersects a meridian once at right angles is called a 'longitude'. A circle on the bounding torus parallel to a meridian is also a meridian and likewise for the longitude. The 'core' of the solid torus is the circle that passes through the center of a meridional disk and is parallel to a longitude.

Let $T_1 = \left\{ (x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 \leq 1/2,\ z^2 + w^2 = 1 - x^2 - y^2 \right\}$.

The set $\left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1/2 \right\}$ describes a 2-dimensional disk of radius $1/\sqrt{2}$. If we fix a point, say (x_0, y_0) in this disk, then $1 - x_0^2 - y_0^2$ is a number r_0^2 with $1/\sqrt{2} \leq r_0 \leq 1$ and the set $\left\{ (z, w) \in \mathbb{R}^2 \mid z^2 + w^2 = r_0^2 \right\}$ is a circle of radius r_0. The set T_1 homeomorphic to $D^2 \times S^1$. Its boundary is the product of a meridian $\left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1/2 \right\}$ and a longitude $\left\{ (z, w) \in \mathbb{R}^2 \mid z^2 + w^2 = 1/2 \right\}$ each with radius $1/\sqrt{2}$. To get cor-

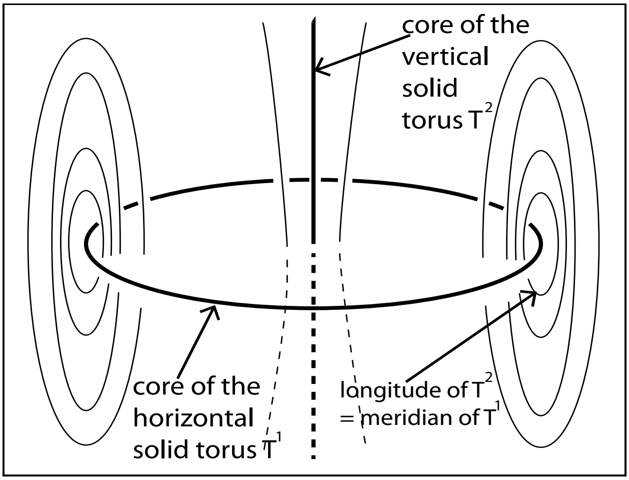

Figure 7. The 3-sphere as a union of solid tori.

responding curves in $\mathbb{R}^4$, we have to fix a point (x_1, y_1, z_1, w_1) with $x_1^2 + y_1^2 = 1/2 = z_1^2 + w_1^2$ and a meridian of T_1 in $\mathbb{R}^4$ is in fact:

$$\left\{(x, y, z_1, w_1) \in \mathbb{R}^4 \,|\, x^2 + y^2 = 1/2\right\}.$$

Likewise, for the longitude. Finally, the core of T_1 is:

$$\left\{(0, 0, z, w) \in \mathbb{R}^4 \,|\, z^2 + w^2 = 1\right\}.$$

The three-sphere is a union of two solid tori.

We really have to 'stretch' our imagination to see a solid torus for which the radius of the central circle is larger than that of a parallel circle on the boundary. Now consider another solid torus:

$$T_2 = \left\{(x, y, z, w) \in \mathbb{R}^4 \,|\, z^2 + w^2 \leq 1/2,\ x^2 + y^2 = 1 - z^2 - w^2\right\}.$$

Solid tori T_1 and T_2 are positioned in $\mathbb{R}^4$ so that a longitude of T_1 bounds a meridional disk of T_2, and a longitude of T_2 bounds a meridional disk for T_1. They intersect in a common boundary, and their union is:

$$T_1 \cup T_2 = \left\{(x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 + w^2 = 1\right\} = S^3 \subset \mathbb{R}^4.$$

Figure 7 is an attempt to describe this in three dimensions. The vertical line along with the point-at-infinity represents the core of T_2.

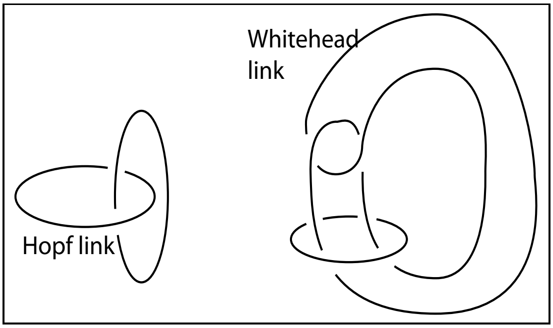

Figure 8. Examples of links with 2 unknotted components.

5. Surgery to Create New Manifolds

In the early 1960s mathematicians Lickorish and Wallace, independently and by different methods, proved that any *closed* (compact without boundary), *orientable*, *connected*, 3-dimensional manifold may be obtained by removing a thickened link from S^3 and filling in the holes differently. This procedure is known as 'surgery'. Links are disjoint unions of knots possibly entangled with one another. See examples in *Figure* 8. Links can have several component circles and these circles may themselves be knotted.

Let's start with the core of T_2 as our 1-component link, and remove the interior (thickened core) of T_2 from S^3. This leaves:

$$T_1 = \left\{ (x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 \leq 1/2,\ z^2 + w^2 = 1 - x^2 - y^2 \right\}.$$

To fill the hole left behind by T_2 with a solid torus T, we now identify a longitude of T with a longitude of T_1 (instead of a longitude with a meridian as done before to obtain S^3), and we identify the boundary of a meridional disk of T with the boundary of a meridional disk of T_1.

We cannot draw the resulting space in three dimensions, but what we can see is that gluing two meridional disks along their boundaries, keeping interiors disjoint, results in a 2-dimensional sphere. A product of this sphere with the longitudinal circle is $S^2 \times S^1$. Techniques from algebraic topology show that this indeed is a new space, not homeomorphic to S^3.

Surgery can be performed on the three-sphere to obtain different 3-manifolds.

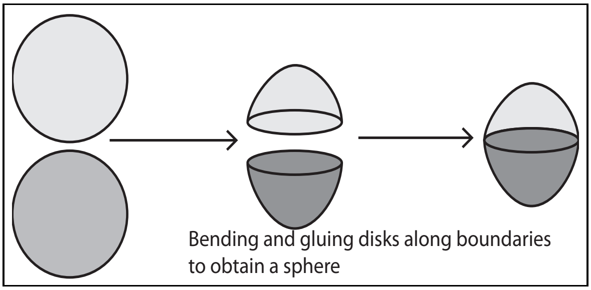

Figure 9. Union of disks with boundary identifications.

The sphere S^3 and the product space $S^1 \times S^2$ are the two ends of a large spectrum of 3-manifolds obtained by surgery on a single unknot.

6. Classification and Invariants

Although knot and link diagrams may be easy to draw, and many of the research problems may be stated simply, a rigorous mathematical treatment takes work. For example, without additional tools, proving that the trefoil and the *Figure* 9 diagrams represent distinct, non-isotopic knots would amount to showing that no sequence of diagram moves can transform one into the other. How do we know that we have exhausted all possible move sequences? And for diagrams with many crossings, even if they happen to represent the same knot, showing this with diagram moves can take forever!

Typically, when mathematicians wish to study a complicated class or category of objects with a class of functions and an established sense of equivalence between two objects, they try to make a transition to an associated category of objects that are easier to compute, distinguish, or study.

Typically, when mathematicians wish to study a complicated class or category of objects (e.g. topological spaces) with a class of functions (e.g. continuous) and an established sense of equivalence (e.g. homeomorphism) between two objects, they try to make a transition to an associated category of objects that are easier to compute, distinguish, or study. Such associated objects may be numbers, polynomials, more complicated algebraic structures, or something else entirely. The association is achieved in such a way that if an object in the original category is transformed into another through an equivalence, then the associated objects are equivalent in the new category as well. Such associated objects are called *invariants*. The minimal number of crossings with which a diagram for a given knot or link can be drawn in the plane is a numerical invariant called the *crossing number*. A link invariant called the *Jones polynomial* helped prove conjectures about crossing numbers that had been open for over a century. Its generalizations resulted in computable invariants of 3-manifolds. Related work that was completed in 1984–87 led to Vaughan Jones and Edward Witten receiving the 1990 Fields Medal, the highest recognition in mathematics.

Knot theory continues to grow as a vibrant area of research within low dimensional topology that is rich with applications. We encourage readers to do an internet search for the latest information and free online resources for knots, links, and their scientific applications. Mathematical basics are covered in the reference books below listed in Suggested Reading: [2, 3] are intended for undergraduate students, whereas [4–6] are written for graduate students with knowledge of point-set and algebraic topology.

The minimal number of crossings with which a diagram for a given knot or link can be drawn in the plane is a numerical invariant called the crossing number.

Suggested Reading

[1] **Unnati Akhouri, Science Smiles, *Resonance*, Vol.23, No.3, pp.249–249, 2017. https://www.ias.ac.in/article/fulltext/reso/023/03/0249-0249.**

[2] **Charles Livingston, Knot Theory, *Carus Mathematical Monographs*, Vol.24, Mathematical Association of America, Washington, DC, 1993.**

[3] **Colin C Adams, *The Knot Book*, Revised reprint of the 1994 original, Published by American Mathematical Society, Providence, RI, 2004.**

[4] **G Burde and H Zieschang, Knots, Second Ed., Vol.5, *de Gruyter Studies in Mathematics*, Walter de Gruyter & Co., Berlin, 2003.**

[5] **L H Kauffman, On Knots, *Annals of Mathematics Studies*, Vol.115, Princeton University Press, Princeton, NJ, 1987.**

[6] **D Rolfsen, Knots and Links, Publish or Perish Inc., Berkeley, Calif., *Mathematics Lecture Series*, No.7, 1976.**

Address for Correspondence
Swatee Naik
Department of Mathematics and Statistics
University of Nevada
Reno, NV 89557
Email: naik@unr.edu