Classroom

In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Fundamental Theorem of Algebra – A Nevanlinna Theoretic Proof

There are several proofs of the fundamental theorem of algebra, mainly using algebra, analysis and topology. The aim of this article is to discuss the fundamental theorem of algebra as an application of Nevanlinna’s second fundamental theorem.

1. Introduction

One of the celebrated theorem in mathematics is the fundamental theorem of algebra, according to which every non constant polynomial over \( \mathbb{C} \) has a root in \( \mathbb{C} \).

Some mathematicians, like Peter Roth (Arithmetica Philosophica, 1608), Albert Girard (L’invention nouvelle en l’Algèbre, 1629), and René Descartes described some version of fundamental theorem of algebra in the early 17th century in their writings.

But the first attempt at proving this theorem was made in 1746 by Jean-Baptiste le Rond d’Alembert; unfortunately his proof was incomplete. Also attempts were made by Euler (1749), de

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Foncenex (1759), Lagrange (1772), and Laplace (1795). But Carl Friedrick Gauss is credited with producing the first correct proof in his doctoral dissertation in 1799, although the proof also had some small gaps.

A rigorous proof was first produced by Argand in 1806 and the first textbook containing the proof is *Cours d’analyse de l’École Royale Polytechnique* (1821) due to Cauchy.

Now a days, there are many proofs of the fundamental theorem of algebra, mainly using algebra, analysis and topology (see [1]).

There are several analytical proofs using complex analysis, for examples, proof based on Liouville’s theorem, Rouche’s theorem, maximum modulus principle [2], Picard’s theorem [3], Cauchy’s integral theorem [4, 5], open mapping theorem [6], power series [7, 8] etc.

The aim of this article is to produce another analytical proof of fundamental theorem of algebra, using Nevanlinna theory.

2. Brief Discussion on Nevanlinna Theory

The theory of meromorphic functions was greatly developed by Rolf Nevanlinna [9] during the 1920s. In both its scope and its power, his approach greatly surpasses previous results, and in his honor the field is now also known as Nevanlinna theory [10].

A function which is regular on the entire complex plane is said to be entire function, where a function is said to be meromorphic if it is analytic on the entire complex plane except possibly at poles.

Let \( f(z) \) be a non-constant meromorphic function and \( a \in \mathbb{C} \cup \{\infty\} \). If \( z_0 \) is a zero of \( f(z) - a \) of multiplicity \( m \), then \( z_0 \) is said to be an \( a \)-point of \( f \) of multiplicity \( m \). If \( a = \infty \), then by \( a \)-point of multiplicity \( m \), we mean a pole of \( f \) of order \( m \).

For \( a \in \mathbb{C} \cup \{\infty\} \), we denote by \( n(r; a; f) \), the number of \( a \)-points of \( f(z) \) in \( |z| \leq r \), where an \( a \)-point is counted according to its multiplicity, and by \( \overline{n}(r; a; f) \), the number of distinct \( a \)-points of
f(z) in the \(|z| \leq r\). Also we define

\[
N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r, \quad (1)
\]

\[
\overline{N}(r, a; f) = \int_0^r \frac{\overline{n(t, a; f)} - \overline{n(0, a; f)}}{t} dt + \overline{n(0, a; f)} \log r, \quad (2)
\]

Next we define

\[
m(r, \infty; f) = m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad (3)
\]

where \(\log^+ x := \max\{\log x, 0\}\) for \(x \geq 0\). Thus \(m(r, f)\) measures the closeness of \(f\) to infinity on \(|z| = r\).

Also, the Nevanlinna’s characteristic function of \(f\) is defined as

\[
T(r, f) = m(r, \infty; f) + N(r, \infty; f).
\]

The basic estimation of Nevanlinna theory is the first fundamental theorem, which states that

For a non-constant meromorphic function \(f(z)\), defined in \(|z| < R\) (\(0 < R \leq \infty\)), and a complex number \(a \in \mathbb{C} \cup \{\infty\},

\[
T \left( r, \frac{1}{f - a} \right) = T(r, f) + O(1),
\]

where \(O(1)\) is a bounded quantity depending on \(f\) and \(a\) but not on \(r\), \(0 < r < R\).

“The term \(m(r, \frac{1}{f - a})\) refers to the average smallness in certain sense of \(f - a\), on the circle \(|z| = r\), the term \(N(r, \frac{1}{f - a})\) refers to the numbers of roots of the equation \(f(z) = a\) in \(|z| < r\). For any \(a\), the sum of these two terms is the same apart from a bounded term.” (see, [10])
Let us now consider an example. Let $P(z) = a_0 z^p + a_1 z^{p-1} + \ldots + a_{p-1} z + a_p$ be a non-constant polynomial function of degree $p \in \mathbb{N}$. Then $P(z) \sim a_0 z^p$ as $r \to \infty$. Thus

$$T(r, P) = m(r, \infty; P) + N(r, \infty; P) = m(r, P) = p \log r + O(1),$$

as $r \to \infty$, where $O(1)$ is a bounded quantity.

Another good estimation of Nevanlinna theory is the second fundamental theorem, which is an extension of Picard’s Little theorem.

Suppose that $f(z)$ is a non-constant meromorphic function in the complex plane and $a_1, a_2, \ldots, a_q$ are $q \geq 2$ distinct values in $\mathbb{C}$. Then

$$(q - 1)T(r, f) < \tilde{N}(r, \infty; f) + \sum_{j=1}^{q} \tilde{N} \left( r, \frac{1}{f - a_j} \right) + S(r, f),$$

where $S(r, f)$ is a quantity which satisfies the condition $\frac{S(r, f)}{T(r, f)} \to 0$ as $r \to +\infty$ outside of a set $E \subset \mathbb{R}^+ \subset (0, \infty)$ whose Lebesgue measure is finite, and $\tilde{N}(r, \infty; f)$ counts poles in ignoring multiplicities.

If there exists a non-constant entire function which does not assume two finite complex numbers, say $a$ and $b$. Then by the second fundamental theorem, we have

$$(2 - 1)T(r, f) < \tilde{N}(r, \infty; f) + \tilde{N} \left( r, \frac{1}{f - a} \right) + \tilde{N} \left( r, \frac{1}{f - b} \right) + S(r, f) = S(r, f),$$

which is impossible. Thus, a non-constant entire function assumes either the whole complex plane or the complex plane minus a single point, which is the Picard’s Little theorem.

3. Proof of the Fundamental Theorem of Algebra

**Lemma 3.1.** A polynomial of the form

$$Q(z) = b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \ldots + b_{m-l} z^l + b_m,$$
has at least one zero, where \( m \) and \( l \) are positive integers satisfying \( m > 2, m > l \geq 2 \) and \( b_i \in \mathbb{C} \) for \( i = 0, 1, 2, \ldots, m - l; m \) with \( b_0 \neq 0 \).

**Proof.** On the contrary, we assume that the polynomial \( Q(z) \) has no zero, then by definition, \( \tilde{N}(r, 0; Q) = 0 \).

Next we define \( F(z) := z^l R(z) \) where \( R(z) = b_0 z^{m-l} + b_1 z^{m-l-1} + b_2 z^{m-l-2} + \ldots + b_{m-l} \). Then

\[
Q(z) = F(z) + b_m, \quad (4)
\]

Thus applying the second fundamental theorem, we have

\[
T(r, F) < \tilde{N}(r, 0; F) + \tilde{N}(r, 0; -b_m; F) + S(r, F) \leq \frac{1}{l} N(r, 0; z^l) + N(r, 0; R(z)) + S(r, Q(z)) + S(r, F) \leq \log r + (m - l) \log r + S(r, F) = \left( \frac{m - l + 1}{m} \right) T(r, F) + S(r, F),
\]

which is absurd. Hence, our assumption is wrong. Thus \( Q(z) \) has at least one zero. Hence the proof.

4. **Proof of the Fundamental Theorem of Algebra**

Assume \( S(n) \) denotes the following statement:

*Any \( n \)-degree polynomial over \( \mathbb{C} \) has at least one zero for any \( n \in \mathbb{N} \).*

Thus it is sufficient to prove that the statement \( S(n) \) is true for all \( n \in \mathbb{N} \), and we have to prove that the statement \( S(n) \) is true using mathematical induction on \( n \).

It is obvious that the statements \( S(1) \) and \( S(2) \) are true, and assume that \( S(k) \) is true for any natural number \( k \geq 3 \). Now we have to show that \( S(k + 1) \) is true.

For this, we consider any polynomial of degree \( k + 1 \) over \( \mathbb{C} \) as

\[
P(z) = a_0 z^{k+1} + a_1 z^k + a_2 z^{k-1} + \ldots + a_k z + a_{k+1},
\]
with $a_0 \neq 0$.

Since the statement $S(k)$ is true, we can find a suitable complex number $h$ such that the coefficient of $z$ in $P(z + h)$ is zero. So by Lemma 3.1, $P(z + h)$ has at least one zero. Consequently, $P(z)$ has at least one zero. Hence the statement $S(k + 1)$ is true.

Thus by mathematical induction, any $n$-degree non-constant polynomial over $\mathbb{C}$ has at least one zero for any $n \in \mathbb{N}$. Hence the proof.

**Suggested Reading**


