# Balanced Number System

## Application to Mathematical Puzzles

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**The article explores the application of binary and ternary number systems to three classical mathematical puzzles – weight problem of Bachet de Méziriac, binary numbers magic trick, and the detection of a counterfeit coin. The article further elucidates the generalization of these puzzles using the balanced number system.**

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### 1. Introduction

We have all grown up with the decimal system of numeration that uses the digits $0, 1, 2, \ldots, 9$. The use of the decimal number system is sometimes attributed to us having ten fingers (counting the thumb) on our hands. But imagine if we were in the animated world of Fred Flintstone or Simpsons where each of the leading characters has four fingers on each hand. Then absurd mathematical expressions like $42 + 36 = 100$ or $42 - 14 = 26$ would not look so absurd. These expressions would be correct if they are calculated in the octal base system. Though the use of numbers in base 10 is the most common in present times, numbers in different bases have also found various applications.

The digital world of computers, mobile phones, audio players, etc., use the binary system (base 2). The ternary numeral system (base 3) finds its application in analog logic where the output of a circuit is either low (grounded state), high, or open (high impedance). The way the Cantor set is constructed, a ternary system is useful for defining it. Soviet computers in the early days of computing incorporated the balanced ternary system that uses $-1, 0, 1$ instead of $0, 1, 2$. Roman numerals are in the biquinary system having 2 and 5 as the base. They have different symbols for $1, 5, 10, 50, 100, 500$ and $1000$. The octal and the hex-

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adecimal systems are also being used in the computers. In the Academy Awards nominated movie *Avatar*, the humanoid species *Na′vi* employs an octal numeral system in their language. Incidentally, they are also shown to have four fingers on each of their hands.

It is not that a numeral system with a base different from 10 has been used only in recent times. Numbers in a base different from 10 have been practiced in ancient times too. While Mayan civilization used numbers in base 20, the Babylonian's number system was a base 60 system.

It is not that a numeral system with a base different from 10 has been used only in recent times. Numbers in a base different from 10 have been practiced in ancient times too. While Mayan civilization used numbers in base 20, the Babylonian's number system was a base 60 system. The duodecimal system or the dozenal system was popular in the seventeenth century. A carpenter's ruler has 12 inches, grocers deal in dozens, there are 12 months in a year and each day is divided into two sets of 12 hours each. F E Andrews [1] lists out a number of advantages of using the duodecimal system in place of the decimal system.

This article deals with how to convert numbers in base 10 to balanced ternary system and application of the balanced ternary system to some mathematical puzzles. The article further explores how these puzzles may be generalized by introducing the balanced *n*-nary system.

## 2. Balanced *n*-nary System

D E Knuth in his monograph *The Art of Computer Programming* comments on the balanced ternary system being the prettiest number system of all. As mentioned above, the balanced ternary number system employs integers $-1, 0, 1$ in place of $0, 1, 2$. It is customary to represent a number in base 2 in terms of 0's and 1's and in base 3 in terms of 0's, 1's and 2's. Any number in base 10 can be represented in base 2 (or base 3) by using the expanded notation. For example:

$(464)_{10} \equiv 1 \times 2^8 + 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^4 = (111010000)_2$

$(464)_{10} \equiv 1 \times 3^5 + 2 \times 3^4 + 2 \times 3^3 + 1 \times 3 + 2 \times 1 = (122012)_3.$

Another form of representing it is by long division where the rightmost column represents the remainder, and the middle row represents the quotient at each step (see *Tables* 1 and 2).

| | | |
|---|---|---|
| 2 | 464 | |
| 2 | 232 | 0 |
| 2 | 116 | 0 |
| 2 | 58 | 0 |
| 2 | 29 | 0 |
| 2 | 14 | 1 |
| 2 | 7 | 0 |
| 2 | 3 | 1 |
| 2 | 1 | 1 |

**Table 1.** Representing 464 in base 2.

| | | |
|---|---|---|
| 3 | 464 | |
| 3 | 154 | 2 |
| 3 | 51 | 1 |
| 3 | 17 | 0 |
| 3 | 5 | 2 |
| 3 | 1 | 2 |

**Table 2.** Representing 464 in base 3.

In ternary system, a non-negative number $n$ may be represented as $3k$ (if divisible by 3) or $3k + 1$ (if it leaves a remainder 1 on dividing by 3) or $3k + 2$ (if it leaves a remainder 2 on dividing by 3), for $k = 0, 1, 2, \ldots$. The number $3k+2$ may also be represented as $3(k+1)-1$. So we could say that instead of leaving a remainder 2 when divided by 3 it leaves a remainder $-1$. Using this idea and performing long division we can have the solution shown in *Table* 3.

Therefore, the balanced ternary representation of 464 in base 3 is $(464)_{10} \equiv (1\text{–}10\text{–}11\text{–}1\text{–}1)_3$.
In expanded form this would represent
$(464)_{10} \equiv 1 \times 3^6 - 1 \times 3^5 - 1 \times 3^3 + 1 \times 3^2 - 1 \times 3 - 1 \times 1.$

Another way of getting this notation from the usual expanded

**Table 3.** Representing 464 in balanced ternary system.

| | | |
|---|---|---|
| 3 | 464 | |
| 3 | 155 | -1 |
| 3 | 52 | -1 |
| 3 | 17 | 1 |
| 3 | 6 | -1 |
| 3 | 2 | 0 |
| 3 | 1 | -1 |

form is:

$$\begin{aligned}(464)_{10} &= 1 \times 3^5 + 2 \times 3^4 + 2 \times 3^3 + 1x3 + 2x1 \\ &= 1 \times 3^5 + (3^5 - 3^4) + (3^4 - 3^3) + 1 \times 3 + (3 - 1) \\ &= 2 \times 3^5 - 1 \times 3^3 + 2 \times 3 - 1 \\ &= (3^6 - 3^5) - 1 \times 3^3 + (3^2 - 3) - 1 \\ &= 1 \times 3^6 - 1 \times 3^5 - 1 \times 3^3 + 1 \times 3^2 - 1 \times 3 - 1\,.\end{aligned}$$

The above calculation illustrates that the number 2 in base 3 can be written as $2 = 1 \times 3 - 1 = (1, -1)_3$. This implies that if the remainder used is $-1$ in place of 2, it is carried forward to the next place value. We may represent it as shown in *Table* 4.

**Table 4.** Carry forward representation in balanced ternary system.

| | $3^6$ | $3^5$ | $3^4$ | $3^3$ | $3^2$ | $3^1$ | 1 |
|---|---|---|---|---|---|---|---|
| Carry forward | 1 | 1 | 1 | 0 | 1 | 1 | |
| 464 in base 3 | 0 | 1 | 2 | 2 | 0 | 1 | 2 |
| 464 in balanced ternary system | 1 | -1 | 0 | -1 | 1 | -1 | -1 |

Use of $-1$ in the representation can become quite cumbersome. Suggested reading [2] represents the integer $-1$ by $T$ whereas [3] represent the ternary digits by $N$ (for $-1$), $Z$ (for 0) and $P$ (for 1). Since the article shall deal with general balanced $n$-nary number system, the representation used will be $L_i$ (if the integer is $i$ places to the left of 0), 0, and $R_i$ (if the integer is $i$ places to the right of

0). Therefore, under this representation:
$(464)_{10} \equiv (R_1, L_1, 0, L_1, R_1, L_1, L_1)_3$.

The concept of balanced number system may be extended for any positive integer $n > 1$. If $n$ is odd, say $2m + 1, m = 1, 2, 3, \ldots$ one can express any natural number $N$ in base $2m + 1$ in terms of $0, L_i, R_i, i = 1, 2, \ldots, m$. As an example let us take another number (*Table* 5):
$(2565)_{10} \equiv (40230)_5 = (R_1, L_1, R_1, L_2, L_2, 0)_5$

**Table 5.** Carry forward representation in balanced base 5.

| | $5^5$ | $5^4$ | $5^3$ | $5^2$ | $5^1$ | 1 |
|---|---|---|---|---|---|---|
| Carry forward | 1 | 0 | 1 | 1 | 0 | |
| 2565 in base 5 | 0 | 4 | 0 | 2 | 3 | 0 |
| 2565 in balanced nary system | 1 | -1 | 1 | -2 | -2 | 0 |
| | $R_1$ | $L_1$ | $R_1$ | $L_2$ | $L_2$ | 0 |

If $n$ is even, say $2m, m = 1, 2, 3, \ldots$ one can express any natural number $N$ in base $2m$ in terms of $0, L_i, R_i, R_m(\text{or } L_m), i = 1, 2, \ldots, m - 1$. We consider another example (*Table* 6):
$(891)_{10} \equiv (31323)_4 = (R_1, L_1, R_2, 0, L_1, L_1)_4$.

**Table 6.** Carry forward representation in balanced base 4.

| | $4^5$ | $4^4$ | $4^3$ | $4^2$ | $4^1$ | 1 |
|---|---|---|---|---|---|---|
| Carry forward | 1 | 0 | 1 | 1 | 1 | |
| 891 in base 4 | 0 | 3 | 1 | 3 | 2 | 3 |
| 891 in balanced quaternary system | 1 | -1 | 2 | 0 | -1 | -1 |
| | $R_1$ | $L_1$ | $R_2$ | 0 | $L_1$ | $L_1$ |

## 3. Application to Weight Problem of Bachet de Méziriac

The weight problem of Bachet de Méziriac requires one to find the minimum number of weights required that can be placed in *either pan* of a balance to weigh any integral number of pounds from 1 to 40.

The weight problem of Bachet de Méziriac, a French mathematician born in the sixteenth century, requires one to find the minimum number of weights required that can be placed in *either pan* of a balance to weigh any integral number of pounds from 1 to 40.

A slight variation of the problem is to find the minimum number of weights required that can be placed in *only one pan* of the balance to weigh any integral number of pounds from 1 to 40. The answer to the former question is four weights with measurements 1 lb, 3 lbs, 9 lbs and 27 lbs, where as the answer to the latter problem is six: 1 lb, 2 lbs, 4 lbs, 8 lbs, 16 lbs and 32 lbs. It is easy to observe that the weights in the first case are powers of 3 and in the second case are powers of 2. So if we need to weigh any integral number of weights from 1 to $n$ we would need weights of measure $1, 3, \ldots, 3k$ such that $n \leq \frac{1}{2}(3^{k+1} - 1)$[1] or $1, 2, \ldots, 2^k$ such that $2^k < n$, for $k = 1, 2, 3, \ldots$. For any given weight measure how do we decide what weights are needed? One way to do is to convert the number in its binary form (for the latter case) or ternary numeral system (for the former case).

[1]For the proof, the reader may refer to [4]. The reader can also refer to [5] and [6].

Let us first consider the case when the weights are placed in only one pan and we wish to measure a weight of 40 lbs.
$40 \equiv (101000)_2$, which can be written in the expanded form as:

$$40 \equiv (101000)_2 = 1 \times 32 + 0 \times 16 + 1 \times 8 + 0 \times 4 + 0 \times 2 + 0 \times 1.$$

Hence one would need a weight of 32 lbs and 8 lbs to measure a weight of 40 lbs.

If the weights can be placed in either of the pan then we convert 40 in its ternary numeral system.

$$40 \equiv (1111)_3 = 1 \times 27 + 1 \times 9 + 1 \times 3 + 1 \times 1\,.$$

This may be interpreted as that in order to measure a weight of 40 lbs, one requires a weight each of 27 lbs, 9 lbs, 3 lbs and 1 lb – four weights in total which is exactly the answer we needed. Let us see what happens if we wish to weigh 50 lbs instead of 40 lbs. As before, we convert 50 in its ternary numeral system.

$$50 \equiv (1212)_3 = 1 \times 27 + 2 \times 9 + 1 \times 3 + 2 \times 1\,.$$

If we interpret it as before, we would need one weight of 27 lbs, two weights of 9 lbs, one weight of 3 lbs and two weights of 1 lb – a total of 6 weights. But this is not the minimum number of weights required. We would need lesser number of weights to measure 50 lbs. If the weights can be kept in either of the two pans, we can look at the problem through the perspective of balanced number system. Let us expand the number 50 using the balanced number system.

$$\begin{aligned} 50 &= 1 \times 27 + 2 \times 9 + 1 \times 3 + 2 \times 1 \\ &= 1 \times 27 + (1 \times 27 - 1 \times 9) + 1 \times 3 + (1 \times 3 - 1 \times 1) \\ &= 2 \times 27 - 1 \times 9 + 2 \times 3 - 1 \times 1 \\ &= (1 \times 81 - 1 \times 27) - 1 \times 9 + (1 \times 9 - 1 \times 3) - 1 \times 1 \\ &= 1 \times 81 - 1 \times 27 - 1 \times 3 - 1 \times 1. \end{aligned}$$

Therefore, one needs four weights – a weight each of 81 lbs, 27 lbs, 3 lbs and 1 lb to measure 50 lbs. In this case, 81 lbs is placed in one pan and 27 lbs, 3 lbs and 1 lb are placed in the other pan.

Similarly, *Table* 4 suggests that in order to measure a weight of 464 lbs, one needs a weight of each 729 lbs and 9 lbs in one pan and weights of 243 lbs, 27 lbs, 3 lbs and 1 lb in the other. Therefore, six weights would be needed. We ask a question here: can this be measured in fewer numbers of weights? Before we answer this question, recall the law of moments in physics that states:

The article by J Madhusudana Rao and T V N Prasanna [7] beautifully explains how certain mathematical problems can be interpreted as physical problems.

*For an object balanced in equillibrium the sum of the clockwise moments is equal to the sum of the anticlockwise moments.*

If the system in *Figure* 1 is in equilibrium, then according to the law of moments, the system may be represented mathematically as:

$\text{Force}_1 \times \text{Distance}_1 = \text{Force}_2 \times \text{Distance}_2.$

The mechanical weighing balance uses the same principal. A mathematical representation of a number in base 3 analogous to the weighing balance is shown in *Figure* 2.

If instead we have a weighing balance that has four pans rather than two (as shown in the *Figure* 3), then in equilibrium, the law of moments will give us: $W_1 \times x_1 + W_2 \times x_2 = W_3 \times x_3 + W_4 \times x_4$.

**Figure 1.** Law of moments.

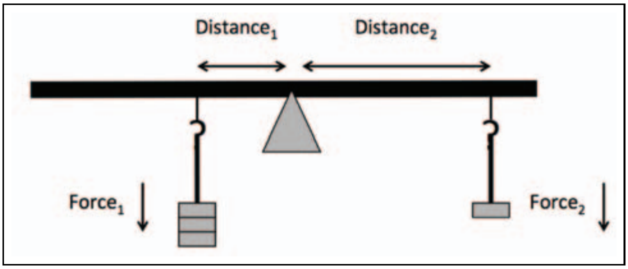



**Figure 2.** Mathematical representation of a number in base 3.

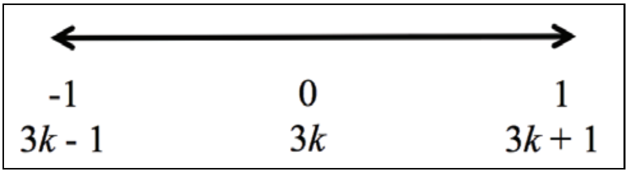



**Figure 3.** Weighing balance with four pans.

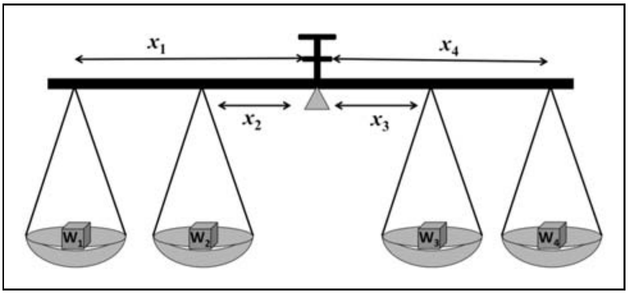
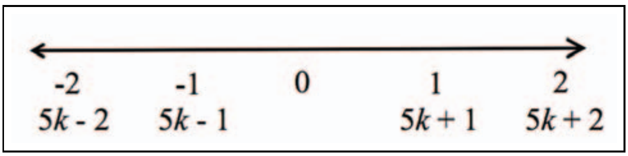



**Figure 4.** Mathematical representation of a number in base 5.




The mathematical representation of a number in base 5 analogous to the weighing balance is shown in *Figure* 4.

To end the section, we again look at the example of weighing 464 lbs in balanced quinary system. Representing 464 in base 5 gives:

$(464)_{10} \equiv (R_1, L_1, L_1, L_2, L_1)_5.$

So, one would need five weights each of denomination 625 lbs, 125 lbs, 25 lbs, 5 lbs and 1 lb. The weights and the quantity ($W$)

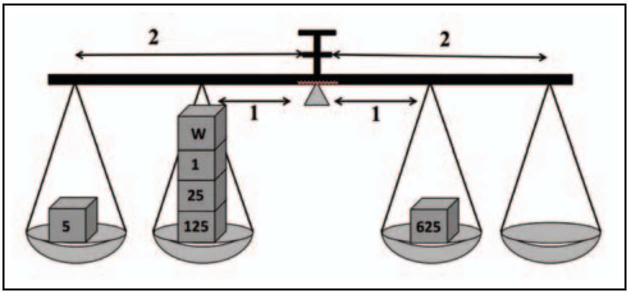



**Figure 5.** Measuring 464 lbs using a weighing balance with four pans.

to be measured will have to be placed as shown in *Figure* 5.

We could also have used four weights of denominations 1, 6, 36 and 216 lbs because,

$464 = 2 \times 216 + 1 \times 36 - 1 \times 6 + 2 \times 1$
or, $(464_{10} \equiv (R_2, R_1, L_1, R_2)_6$.

In fact, we can measure all integral weights upto 259 lbs using these four weights, but we shall need a weighing balance with six pans! Decrease in the number of weights being used increases the number of pans in the weighing balance.

## 4. Application to Binary Numbers Magic Trick

We now explore the usage of balanced number system in another popular mathematical trick – the binary numbers magic trick. The trick requires a volunteer to think of any number between 1 and 30. The person performing the trick asks the volunteer to point out the cards from the set of five cards in *Table* 7 (*Figure* 6).

How did the performer guess the number? One can notice that the sum of the top left corner number of the selected cards adds up to 27. That is exactly what the performer does. But we wish to understand why the trick works. Please note that variation of the trick exists with a different sets of cards.

The trick is very much like the weight problem where the weight is to be kept in only one pan. If one carefully looks at the number in each card and convert it to binary number, one will get a fair idea of how the trick works. The 0's and 1's appearing in the

| | | | |
|---|---|---|---|
| 1 | 3 | 5 | 7 |
| 9 | 11 | 13 | 15 |
| 17 | 19 | 21 | 23 |
| 25 | 27 | 29 | 31 |

(a)

| | | | |
|---|---|---|---|
| 2 | 3 | 6 | 7 |
| 10 | 11 | 14 | 15 |
| 18 | 19 | 22 | 23 |
| 26 | 27 | 30 | 31 |

(b)

| | | | |
|---|---|---|---|
| 4 | 5 | 6 | 7 |
| 12 | 13 | 14 | 15 |
| 20 | 21 | 22 | 23 |
| 28 | 29 | 30 | 31 |

(c)

| | | | |
|---|---|---|---|
| 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 |
| 24 | 25 | 26 | 27 |
| 28 | 29 | 30 | 31 |

(d)

| | | | |
|---|---|---|---|
| 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 |
| 28 | 29 | 30 | 31 |

(e)

**Table 7.** Binary magic using cards.

binary equivalent of the number allows you to know in which card the number should be placed and in which it should not be.

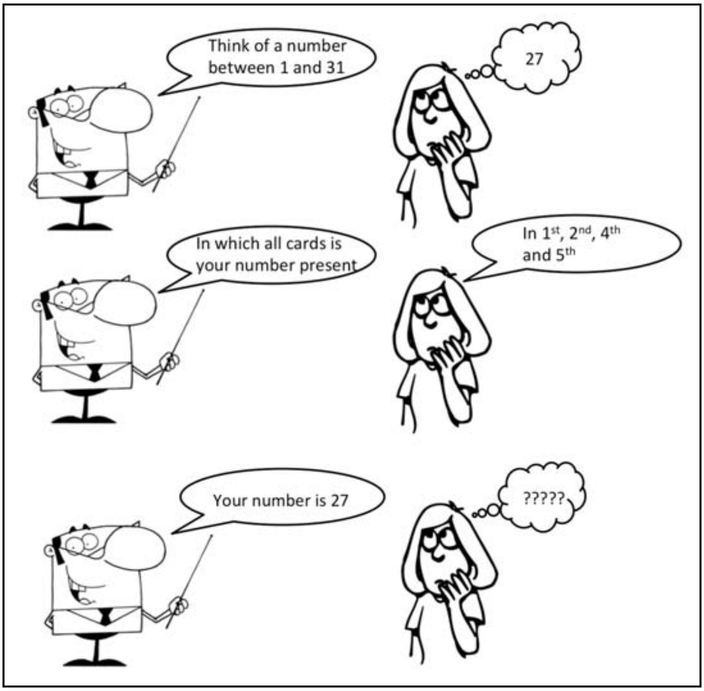



**Figure 6.** Binary magic card trick.

For example if one looks at 27 whose binary representation is $(11011)_2$, then this tells us that 27 must be placed in the first, second, fourth and fifth card and should not be there in the third card.

If one wishes to generalize the trick using the ternary system, one needs to go back to the weighing problem in ternary system or to be more precise balanced ternary system. If each number from 1 to 80 is expressed in its balanced ternary notation, then the cards in *Table* 8 can be used to guess any number from 1 to 80. The only alteration is that now we use two colors. This is analogous to using the two pans – left and right in the weight problem. If the number is in the first card and is black add 1 and if it is red subtract 1. Similarly, if the number is in the second card and is black add 3 and if it is red subtract 3. For the third card the cue number is 9 where as for the fourth card it is 27. The resultant answer is the required number. For numbers greater than 40 the resultant sum will come out to be negative.

If we look at the number 58, for example:

$(58)_{10} \equiv (R_1, L_1, 0, R_1, R_1)_3$.

The number 58:

- is present in the first card and is black. Therefore add 1.
- is also there in the second card and is black; add 3 so that the resultant sum is 4.
- is not there in the third card but present in the fourth card in red color. Hence, subtract 27 so that the resultant is −23.

If there was a fifth card, its cue number would be 81 and 58 would be present in it in black colour. Therefore we add 81 to the resultant to get $58 = 81 - 23$.

Recall that in base 5, one needed a weighing balance with four pans. So if the trick is to be extended one would need four colors (*Table* 9). If the number is present in the first card add 1 – the cue number, if it is black, two times 1 if it is blue, −2 times 1 if it is

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 7 | 8 |
| 10 | 11 | 13 | 14 | 16 | 17 |
| 19 | 20 | 22 | 23 | 25 | 26 |
| 28 | 29 | 31 | 32 | 34 | 35 |
| 37 | 38 | 40 | 41 | 43 | 44 |
| 46 | 47 | 49 | 50 | 52 | 53 |
| 55 | 56 | 58 | 59 | 61 | 62 |
| 64 | 65 | 67 | 68 | 70 | 71 |
| 73 | 74 | 76 | 77 | 79 | 80 |

(a)

| | | | | | |
|---|---|---|---|---|---|
| 2 | 5 | 11 | 14 | 20 | 23 |
| 3 | 6 | 12 | 15 | 21 | 24 |
| 4 | 7 | 13 | 16 | 22 | 25 |
| 29 | 32 | 38 | 41 | 47 | 50 |
| 30 | 33 | 39 | 42 | 48 | 51 |
| 31 | 34 | 40 | 43 | 49 | 52 |
| 56 | 59 | 65 | 68 | 74 | 77 |
| 57 | 60 | 66 | 69 | 75 | 78 |
| 58 | 61 | 67 | 70 | 76 | 79 |

(b)

| | | | | | |
|---|---|---|---|---|---|
| 5 | 14 | 32 | 41 | 59 | 68 |
| 6 | 15 | 33 | 42 | 60 | 69 |
| 7 | 16 | 34 | 43 | 61 | 70 |
| 8 | 17 | 35 | 44 | 62 | 71 |
| 9 | 18 | 36 | 45 | 63 | 72 |
| 10 | 19 | 37 | 46 | 64 | 73 |
| 11 | 20 | 38 | 47 | 65 | 74 |
| 12 | 21 | 39 | 48 | 66 | 75 |
| 13 | 22 | 40 | 49 | 67 | 76 |

(c)

| | | | | | |
|---|---|---|---|---|---|
| 14 | 41 | 23 | 50 | 32 | 59 |
| 15 | 42 | 24 | 51 | 33 | 60 |
| 16 | 43 | 25 | 52 | 34 | 61 |
| 17 | 44 | 26 | 53 | 35 | 62 |
| 18 | 45 | 27 | 54 | 36 | 63 |
| 19 | 46 | 28 | 55 | 37 | 64 |
| 20 | 47 | 29 | 56 | 38 | 65 |
| 21 | 48 | 30 | 57 | 39 | 66 |
| 22 | 49 | 31 | 58 | 40 | 67 |

(d)

**Table 8.** Ternary magic cards.

red and −1 times 1 if it is green. If the number is present in the second card, add 5 (the cue number) if it is black, 2 times 5 if it is blue, −2 times 5 if it is red and −1 times 5 if it is green. For the third card the procedure remains the same but the cue number is now 25. Till 62 ($62 = 2 \times 1 + 2 \times 5 + 2 \times 25$), one would get a positive sum. Beyond 62, till 124 ($< 125$), one would get a negative number. Add the negative number to 125 to get the answer.

In each set of cards a number of patterns can be observed. The reader may like to work out these patterns to generalize the puzzle. They may also refer to [8].

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 6 | 7 | 8 | 9 |
| 11 | 12 | 13 | 14 |
| 16 | 17 | 18 | 19 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 111 | 112 | 113 | 114 |
| 116 | 117 | 118 | 119 |
| 121 | 122 | 123 | 124 |

(a)

| | | | |
|---|---|---|---|
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |
| 4 | 10 | 15 | 20 |
| 5 | 11 | 16 | 21 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 105 | 110 | 115 | 120 |
| 106 | 111 | 116 | 121 |
| 107 | 112 | 117 | 122 |

(b)

| | | | |
|---|---|---|---|
| 13 | 38 | 63 | 88 |
| 14 | 39 | 64 | 89 |
| 15 | 40 | 65 | 90 |
| 16 | 41 | 66 | 91 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 35 | 60 | 85 | 110 |
| 36 | 61 | 86 | 111 |
| 37 | 62 | 87 | 112 |

(c)

**Table 9.** Quinary magic cards.

## 5. Detecting a Counterfeit Coin

A classical mathematical puzzle requires a *fake coin* to be detected from a pile of twelve coins that are identical in appearance. The fake coin may either be lighter or heavier. The problem is to determine the minimum number of weighings needed to detect the *fake coin* using a balance scale without the measuring scales. The problem is stated as follows in the literature:

A classical mathematical puzzle requires a *fake coin* to be detected from a pile of twelve coins that are identical in appearance. The fake coin may either be lighter or heavier. The problem is to determine the minimum number of weighings needed to detect the *fake coin* using a balance scale without the measuring scales.

*There are $m \geq 3$coins identical in appearance. Only one coin differs from the others in weight, though it is not known in which direction. What is the smallest number of weighings needed to find this coin and identify its type using balance scales without measuring weights?*

J Sarkar and B K Sinha [9] constructed both sequential and non-sequential weighing plans for minimum number of weighings needed to detect at most one fake coin for any given number of coins. In this article, generalization of the Dyson's method presented in [10] for a balance with four pans is presented (*Figure* 7).

The number of coins $m$ that can be detected in $n$ weighings using a weighing balance with four pans will now satisfy:

$$m \leq \tfrac{1}{2}(5^n - 5).$$

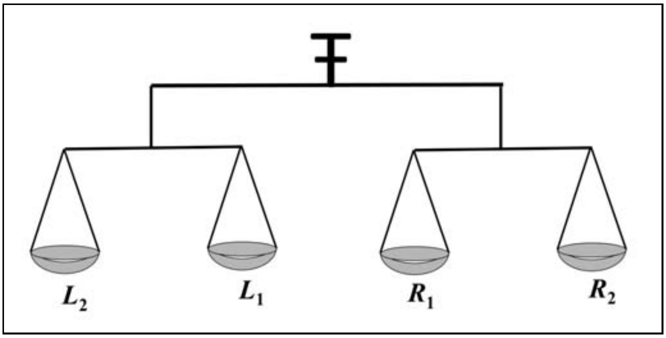



**Figure 7.** Detecting a fake coin using a weighing balance with four pans.

The argument is similar to that presented in [7]. We illustrate the use of balanced number system for the special case when $n = 3$ and $m = 60$. The coins are marked using the balanced numbers $-2, -1, 0, 1, 2$ instead of $0, 1, 2, 3, 4$. The *right markers* are obtained using the cyclic permutations and the *left markers* corresponding to each *right markers* are such that the sum is zero. The *right markers* and the *left markers* of each of the sixty coins is given *Table* 10.

The coins are divided into five sets $M(i, -2)$, $M(i, -1)$, $M(i, 0)$, $M(i, 1)$ and $M(i, 2)$ for $1 \leq i \leq 3$. The use of the balance quinary system aligns to our understanding of integers – negative 2 implies 2 units to the left and positive 2 suggests 2 units on the right. Therefore, in the first weighing ($i = 1$) coins in the set $M(i, -2)$ are placed in the pan $L_2$, coins in the set $M(i, -1)$ are placed in the pan $L_1$, coins in the set $M(i, 1)$ are placed in the pan $R_1$ and coins in the set $M(i, 2)$ are placed in the pan $R_2$. In case the left side is heavier, the weighing is denoted by the digit $-2$ or $-1$ depending upon whether the pan $L_2$ or $L_1$ is heavier. Similarly if the right side is heavier, the weighing is denoted by the digit 1 or 2 depending upon whether the pan $R_1$ or $R_2$ is heavier. If the pans balance, the weighing is denoted by 0. The combination of the digits obtained from the three weighing gives the marker of the coin.

As an example, let us assume that the pan $R_1$ on right side is heavier (*Figure* 8). Then the digit for the first weighing is 1.

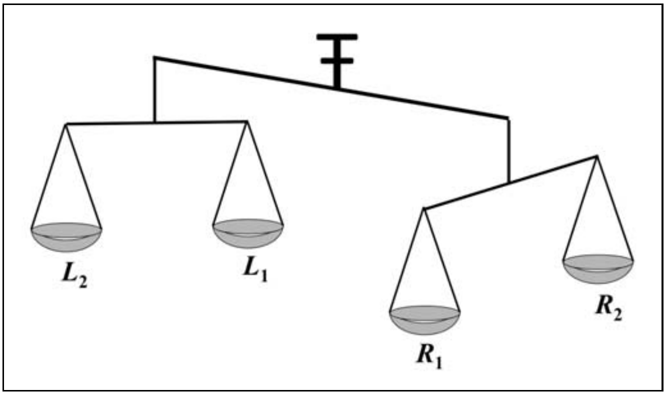
| Coin No. | Left Marker | Right Marker |
|---|---|---|
| 1 | 2, 2, 1 | −2, −2, −1 |
| 2 | 2, 2, 0 | −2, −2, 0 |
| 3 | 2, 1, 2 | −2, −1, −2 |
| 4 | 2, 1, 1 | −2, −1, −1 |
| 5 | 2, 1, 0 | −2, −1, 0 |
| 6 | 2, 1, −1 | −2, −1, 1 |
| 7 | 2, 1, −2 | −2, −1, 2 |
| 8 | 2, 0, 2 | −2, 0, −2 |
| 9 | 2, 0, 1 | −2, 0, −1 |
| 10 | 2, 0, 0 | −2, 0, 0 |
| 11 | 2, 0, −1 | −2, 0, 1 |
| 12 | 2, 0, −2 | −2, 0, 2 |
| 13 | 1, 1, 0 | −1, −1, 0 |
| 14 | 1, 1, −1 | −1, −1, 1 |
| 15 | 1, 0, 2 | −1, 0, −2 |
| 16 | 1, 0, 1 | −1, 0, −1 |
| 17 | 1, 0, 0 | −1, 0, 0 |
| 19 | 1, 0, −2 | −1, 0, 2 |
| 20 | 1, −1, 2 | −1, 1, −2 |
| 21 | 1, −1, 1 | −1, 1, −1 |
| 22 | 1, −1, 0 | −1, 1, 0 |
| 23 | 1, −1, −1 | −1, 1, 1 |
| 24 | 1, −1, −2 | −1, 1, 1 |
| 25 | 0, 0, −1 | 0, 0, 1 |
| 26 | 0, 0, −2 | 0, 0, 2 |
| 27 | 0, −1, 2 | 0, 1, −2 |
| 28 | 0, −1, 1 | 0, 1, −1 |
| 29 | 0, −1, 0 | 0, 1, 0 |
| 30 | 0, −1, −1 | 0, 1, 1 |

| Coin No. | Left Marker | Right Marker |
|---|---|---|
| 31 | 0, −1, −2 | 0, 1, 2 |
| 32 | 0, −2, −2 | 0, 2, 2 |
| 33 | 0, −2, 1 | 0, 2, −1 |
| 34 | 0, −2, 0 | 0, 2, 0 |
| 35 | 0, −2, −1 | 0, 2, 1 |
| 36 | 0, −2, −2 | 0, 2, 2 |
| 37 | −1, 2, 2 | 1, −2, −2 |
| 38 | −1, 2, 1 | 1, −2, −1 |
| 39 | −1, 2, 0 | 1, −2, 0 |
| 40 | −1, 2, −1 | 1, −2, 1 |
| 41 | −1, 2, −2 | 1, −2, 2 |
| 42 | −1, −1, 2 | 1, 1, −2 |
| 43 | −1, −1, −2 | 1, 1, 2 |
| 44 | −1, −2, 2 | 1, 2, −2 |
| 45 | −1, −2, 1 | 1, 2, −1 |
| 46 | −1, −2, 0 | 1, 2, 0 |
| 47 | −1, −2, −1 | 1, 2, 1 |
| 49 | −2, 2, 2 | 2, −2, −2 |
| 50 | −2, 2, 1 | 2, −2, −1 |
| 51 | −2, 2, 0 | 2, −2, 0 |
| 52 | −2, 2, −1 | 2, −2, 1 |
| 53 | −2, 2, −2 | 2, −2, 2 |
| 54 | −2, 1, 2 | 2, −1, −2 |
| 55 | −2, 1, 1 | 2, −1, −1 |
| 56 | −2, 1, 0 | 2, −1, 0 |
| 57 | −2, 1, −1 | 2, −1, 1 |
| 58 | −2, 1, −2 | 2, −1, 2 |
| 59 | −2, −2, 2 | 2, 2, −2 |
| 60 | −2, −2, 1 | 2, 2, −1 |

**Table 10.** Coin markers.

In the second weighing assume that the pan $L_2$ on the left side is heavier. Then the digit corresponding to the second weighing is $-2$. In the third weighing if all the four pans balance, then the digit is 0. Combining the three we get the marker $1, -2, 0$

**Figure 8.** Weighing for the counterfeit coin.




which is the *right marker* for coin number 39. Hence, we detect coin number 39 to be fake which is heavier than the rest (also see [11]).

## Suggested Reading

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