What Exactly is the Electric Field at the Surface of a Charged Conducting Sphere?

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In the presently available literature, one finds distinct results for the electric field at the surface of a charged conducting sphere. In most textbooks, only a simple model is presented in which the electric field leaps from zero (inside the sphere) to a maximum value (just outside the sphere), as follows from Gauss's law. For points exactly at the surface, the charge surrounded by the Gaussian surface becomes ambiguous, and this law is inconclusive. In this paper, by treating the spherical surface as a series of rings, it is shown that that field evaluates to half the discontinuity mentioned above, a result which agrees with more elaborate microscopic models.

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1. Introduction

In physics, while seeking a deeper understanding of a topic, we often formulate simple questions which, though unrealizable from an experimental point of view, could still be satisfactorily answered. In this note, I bring such a topic into the open, namely the electric field in a point located exactly at the surface of a charged conducting sphere in electrostatic equilibrium. Of course, it is impossible in practice to build a perfectly-shaped sphere. Even the surface of a well-polished pure metal, which seems smooth to the naked eye, reveals a high degree of roughness when observed under a microscope. The metallic surface is at an average distance R of the center, with a thickness of hundreds of atoms. Despite the practical impossibility of saying whether a point belongs or not to the surface, I shall determine the electric field at the surface of an ideal, perfect sphere.

Keywords
Electric field, charged conductors, Gauss's law, half-factor, charged sphere, electrostatics.

2. Results and Justifications Found in Textbooks

The electric field at the surface of a spherical conductor is a typical theme for which one finds distinct presentations in textbooks. For instance, in Giancoli's high school level textbook (Exercise 16–12), one finds the following *incomplete* presentation for the field of a spherical shell of radius r_0 uniformly charged [1][1]:

[1]This example actually treats of the electric field created by a *spherical shell uniformly charged*, but Gauss's law guarantees that this field is the same of that from a conducting sphere.

> "Because the charge is distributed symmetrically, the electric field must be *symmetric*. Thus the field *outside* the shell must be directed radially outward (inward if $Q < 0$) and must depend only on r. (...) Thus
>
> $$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, \quad r > r_0 . \tag{1}$$
>
> (...) *Inside* the shell, the field must also be symmetric. So E must again have the same value at all points on a spherical gaussian surface concentric with the shell. Thus, E can be factored out of the sum and, with $Q_{\text{enclosed}} = 0$ since the charge inside the surface is zero, we have:
>
> $$E = 0, \quad r < r_0 , \tag{2}$$
>
> inside a uniform spherical shell of charge."

Here, r is the distance from the center of the shell, Q is the electric charge on the shell, and ϵ_0 is the permittivity of free space. Note that the field for a point *exactly at the surface* of the shell (i.e. for $r = r_0$) is not mentioned. *Incomplete* presentations like this appear in many textbooks for undergraduates [2, 3, 4].

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In other textbooks, an *incorrect* result is deduced from an erroneous application of Gauss's law. For instance, in Halliday *et al.* [5], one of the most adopted introductory physics textbooks, one finds:

$$E(r) = \begin{cases} 0, & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, & r \geq R . \end{cases} \tag{3}$$

Now, the field *at* the surface is explicitly mentioned:

$$E(R) = \frac{1}{4\pi\epsilon_0}\,\frac{Q}{R^2}\,. \tag{4}$$

This is also the result presented in a popular textbook by Young and Freedman [6].

In Section 24.4 of Serway and Beichner's textbook [2], though the case $r = R$ is not mentioned, the case of points *just outside* a charged conducting sphere is treated within the Gauss's law formalism, in which a pillbox enclosing a point at the surface is taken into account to show that:

$$E_{\text{just out}} = \frac{4\pi \mathrm{k}\, q}{A_{\text{out}}} = \frac{\mathrm{k}\, Q}{R^2}\,, \tag{5}$$

where $\mathrm{k} = 1/(4\pi\epsilon_0)$ is the Coulomb constant. In an old textbook by Kip [7], this correct result is discussed in details, but his equation (2.8) implies that:

$$E(R) = \frac{1}{4\pi\epsilon_0}\,\frac{Q}{R^2}\,, \tag{6}$$

a result that is complemented with the following comment (see p.51 of [7]):

> "Thus the field *at* the surface of the shell is exactly the same as though all the charge Q were located at the center of the sphere defined by the shell."

Therefore, the strength of the electric field at the surface should be $\mathrm{k}Q/R^2$.

In most texts covering this topic, one finds either incomplete or incorrect results, but in an old textbook by Purcell one finds the following unsettling result [8]:

$$E(R) = 2\pi\,\mathrm{k}\,\sigma = \frac{1}{2}\,\frac{\mathrm{k}\,Q}{R^2}\,, \tag{7}$$

where $\sigma = Q/\left(4\pi R^2\right)$ is the charge density on the surface of the sphere. Here is a result that certainly has caught people unaware, mainly those who follow faithfully their preferred textbook. If so, they will find it startling that, though the derivation presented in [8] is invalid, the result is indeed *correct*! Curiously, this 'half-result' can be found in some textbooks directed at high school students in Brazil, [9, 10, 11], but they do not present any justificative (not even a reference)[2].

[2]Since the first and the second editions of the book by Purcell were largely adopted in courses of electricity and magnetism for undergraduates in Brazil from the early 1970s to the end of the 1990s, I suspect that it can be the source for the factor $\frac{1}{2}$ found in Brazilian textbooks, though it also appears in an older textbook by Bruhat, who gives only a qualitative proof [12].

Purcell [8], begins by using Gauss's law to show that the field inside a spherical shell with uniform charge density σ is null, whereas the field *just outside* the shell is $4\pi\mathrm{k}\sigma$ $(= \mathrm{k}Q/R^2)$, as seen in (5). For points at the surface, he invokes Newton's third law to show that the force exerted on a small patch of area $\mathrm{d}A$ due to the charges $\mathrm{d}q = \sigma\,\mathrm{d}A$ within the patch itself is null, which simplifies the problem, since it allows him to take into account all the charges on the entire surface of the shell to calculate the force on the patch. He then argues that the field at the surface is given by the average:

$$E(R) = \frac{E(R^-) + E(R^+)}{2} = \frac{0\ + 4\pi\mathrm{k}\sigma}{2} = 2\pi\mathrm{k}\,\sigma = \frac{1}{2}\,\frac{\mathrm{k}Q}{R^2}\,. \qquad (8)$$

Purcell justifies this averaging by analyzing the field created by a layer of charge with a finite thickness $x_0 > 0$. E_1 being the field at the inner surface $(x = 0)$ and E_2 the field just outside, at $x = x_0$, it follows from Gauss's law that:

$$E_2 - E_1 = 4\pi\mathrm{k}\,\sigma. \qquad (9)$$

He then assumes that the field changes continuously from E_1 to E_2, according to an unknown charge density $\rho(x)$. For a much thinner slab, of thickness $\mathrm{d}x \ll x_0$, which encloses a charge $\rho(x)\,\mathrm{d}x$ per unit area, the force *per unit area* is $\mathrm{d}f = E(x)\rho(x)\,\mathrm{d}x$, and hence the total force per unit area is $f = \int_0^{x_0} E(x)\,\rho(x)\,\mathrm{d}x$. Once more, Gauss's law is applied in the substitution of $4\pi k\,\rho(x)\,\mathrm{d}x$ by $\mathrm{d}E$. This leads him to:

$$f = \frac{1}{4\pi\mathrm{k}} \int_{E_1}^{E_2} E\ \mathrm{d}E = \frac{1}{8\pi\mathrm{k}}\left({E_2}^2 - {E_1}^2\right), \qquad (10)$$

which, from (9), simplifies to

$$f = \frac{1}{2}\,(E_2 + E_1)\,\sigma\,. \qquad (11)$$

For a spherical shell, this gives $E(R) = f/\sigma = \frac{1}{2}\left[E(R^+) + E(R^-)\right] = 2\pi k\sigma$, as he wanted to show. He then claims that this justificative should remain valid for any x_0, even in the case of an ideal spherical shell (with a null thickness), but this is an *invalid* argument because the finiteness of x_0 impedes us to take $x_0 = 0$. The procedure of taking thin slabs with thickness $\mathrm{d}x \ll x_0$ in order to develop the integration in (10) is impossible for $x_0 = 0$, which is just the case of a charged spherical shell. Therefore, Purcell's justificative for the half-factor based upon Gauss's law does not hold after a careful mathematical inspection[3].

[3] In a more recent edition (see [13]), Purcell and Morin present the same justificative.

3. Mathematical Validation of the 'Half-factor'

Let us derive the correct result mentioned in (7) by another procedure, *without applying Gauss's law*.

Theorem 1. *For an uniformly-charged spherical shell with radius R and charge Q in electrostatic equilibrium, the electric field at a point P on its surface is given by:*

$$\mathbf{E}_P = \frac{1}{2}\,\frac{\mathrm{k}Q}{R^2}\,\hat{\mathbf{r}}\,. \tag{12}$$

Proof. Let P be the point of intersection of the spherical shell with the z-axis, at $z = R$, at the top of the sphere, as illustrated in *Figure* 1. By cutting the shell to a large number of horizontal, thin circular ribbons of charge $\mathrm{d}Q$ and radius r, from the azimuthal symmetry of the ribbons, it is clear that the electric field $\mathbf{E}_P$ points along the z-axis direction. So,

$$\mathbf{E}_P = \left(\int_{\text{ribbons}} \mathrm{d}E_z\right)\hat{\mathbf{z}}\,, \tag{13}$$

where $\mathrm{d}E_z$ is the field created at point P by the horizontal ribbon centered at point $(0, 0, z)$, which lies at a distance $R - z$ from P. From the well-known electric field of a uniformly-charged ring of radius r (see equations 22–16 in [5]), one has

$$\mathrm{d}E_z = \mathrm{k}\,\frac{(R - z)}{\left[\,r^2 + (R - z)^2\right]^{\frac{3}{2}}}\,\mathrm{d}Q\,. \tag{14}$$

The charge distribution being uniform, then $\mathrm{d}Q = \sigma\,\mathrm{d}A = \sigma\, 2\pi\, r\,\mathrm{d}s$, where $\mathrm{d}s$ is the width of the ribbon. Since $r^2 = R^2 - z^2 = (R+z)\,(R-z)$, one finds:

$$\mathrm{d}E_z = 2\pi\,\mathrm{k}\,\sigma\,\frac{(R-z)\,r\,\mathrm{d}s}{(2R^2 - 2R\,z)^{\frac{3}{2}}} = 2\pi\,\mathrm{k}\,\sigma\,\frac{(R-z)\,\sqrt{R^2-z^2}}{(2R)^{\frac{3}{2}}\,(R-z)^{\frac{3}{2}}}\,\mathrm{d}s$$
$$= 2\pi\,\mathrm{k}\,\sigma\,\frac{\sqrt{R+z}}{(2R)^{\frac{3}{2}}}\,\mathrm{d}s\,. \quad (15)$$

On substituting $\mathrm{d}s = R\,\mathrm{d}\tilde{\theta}$, where $\tilde{\theta} = \pi/2 - \theta$ is the complimentary of the polar angle θ (see *Figure* 1), one has:

$$\mathrm{d}E_z = 2\pi\,\mathrm{k}\,\frac{Q}{4\pi\,R^2}\,\frac{\sqrt{R+z}}{(2R)^{\frac{3}{2}}}\,R\,\mathrm{d}\tilde{\theta} = \mathrm{k}\,\frac{Q}{2R^2}\,\frac{\sqrt{R+z}}{(2R)^{\frac{3}{2}}}\,\frac{\mathrm{d}z}{\cos\tilde{\theta}}\,, \quad (16)$$

where the last factor comes from the differentiation of $\sin\tilde{\theta} = z/R$. Therefore:

$$\mathrm{d}E_z = \mathrm{k}\,\frac{Q}{2R^2}\,\frac{\sqrt{R+z}}{2R\,\sqrt{2R}}\,\frac{\mathrm{d}z}{\sqrt{1 - z^2/R^2}} = \mathrm{k}\,\frac{Q}{4R^2\,\sqrt{2R}}\,\frac{\mathrm{d}z}{\sqrt{R-z}}\,. \quad (17)$$

The integration for all ribbons composing the shell yields:

$$E_z = \int_{\text{ribbons}} \mathrm{d}E_z = \mathrm{k}\,\frac{Q}{4R^2\,\sqrt{2R}}\int_{-R}^{+R}\frac{\mathrm{d}z}{\sqrt{R-z}}\,. \quad (18)$$

At this point, it could be argued that, according to Coulomb's law, the electric field contribution from the element of charge $\mathrm{d}Q$ corresponding to the 'last' ring centered at $z = R$ would be infinite, which could invalidate our procedure. Let us show that this is not the case by solving the above *improper integral* carefully, within the rigour of calculus. From (18), one has:

$$E_z = \mathrm{k}\,\frac{Q}{4R^2\,\sqrt{2R}}\,\lim_{\varepsilon\to 0^+}\int_{-R}^{R-\varepsilon}\frac{\mathrm{d}z}{\sqrt{R-z}}\,. \quad (19)$$

Since, apart from an arbitrary constant, $\int 1/\sqrt{R-z}\,\mathrm{d}z = -2\sqrt{R-z}$, then

$$E_z = -\,\mathrm{k}\,\frac{Q}{2R^2\,\sqrt{2R}}\,\lim_{\varepsilon\to 0^+}\left[\sqrt{R-(R-\varepsilon)} - \sqrt{R-(-R)}\,\right]$$
$$= -\,\mathrm{k}\,\frac{Q}{2R^2\,\sqrt{2R}}\,\lim_{\varepsilon\to 0^+}\left(\sqrt{\varepsilon} - \sqrt{2R}\,\right)$$
$$= \frac{1}{2}\,\frac{\mathrm{k}Q}{R^2}\,, \quad (20)$$

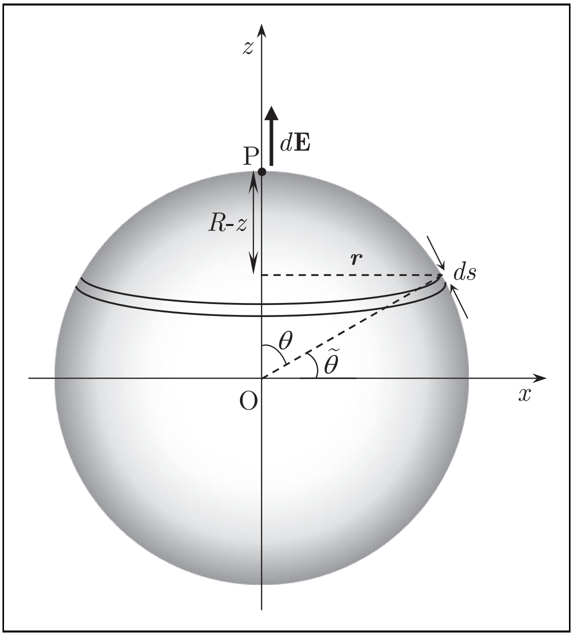

Figure 1. A spherical shell with radius R, uniformly charged with a density $\sigma = Q/(4\pi R^2)$. The shell is cut into a large number of thin horizontal ribbons, each having a charge $\mathrm{d}Q = \sigma\,\mathrm{d}A$. This charge creates an electric field $\mathrm{d}\mathbf{E}$ at a point P on the shell surface (for simplicity, we are assuming $Q > 0$). The field at P is calculated by treating each ribbon as an uniformly charged ring of radius r, with $0 \leq r \leq R$.

as we wanted to show. □

Since only *macroscopic* models make sense in classical electrostatics, the mathematical proof of the existence of the half-factor, above, seems to put an end to the discussion about the electric field on the surface of a conductor sphere. Interestingly, this half-factor arises even in more elaborate *microscopic*, quantum-mechanical models in which both the factors – the distribution of free charge and the field – change continuously when we pass from points inside to just outside a charged conductor [14]. According to these models, a thin transition slab with extension of a few atoms is formed in which the electric field increases smoothly from nearly 0 (inside the conductor) to its maximum value σ/ϵ_0 (at a point about 4 Å outside the conductor), as pointed out in *Figure* I.5 of [14]. In the figure[4], clearly the field in a point *at* the surface (i.e. at $x = 0$) is *half the maximum*, i.e. $\sigma/(2\,\epsilon_0)$.

Interestingly, half-factor arises even in elaborate *microscopic*, quantum-mechanical models in which both the factors – the distribution of free charge and the field – change continuously when we pass from points inside to just outside a charged conductor.

[4]This shows that the 'half-factor' has a meaning at the microscopic level, contrarily to what is argued by Gaspar in Exercise 6, on p.41 of [15].

In a recent paper [16], Assad argues that the half-factor is not supported by any plausible justificative and that the electric field at the surface of a conducting sphere cannot even be defined because there are source charges at the surface. However, we have presented, above, an integration procedure which yields just that half-factor! With regard to the presence of charges on the surface, this is not a problem in itself because the field at a point inside a dielectric sphere of radius R uniformly-charged is given by $\mathrm{k}\, Q\, r/R^3$, valid for all $r \leq R$, as seen, for e.g., at p.676 of [5]. So we have an electric field even at points where source charges are present. Indeed, his other argument that the potential function is not differentiable at $r = R$ does not imply that the field is undefined there, this only implies that the field cannot be described by a continuous function of r, there at $r = R$. In other words, the lateral derivatives of the potential are distinct at the surface, but this says nothing about the field at $r = R$, because the potential is not represented by a differentiable function there.

4. Conclusion

For a charged conducting sphere, Gauss's law promptly yields a well-known discontinuity in the electric field, as stated in (1) and (2). For a point at the surface of the sphere, one finds distinct results in literature, including some textbooks in which a 'half-factor' multiplies the maximum strength $\mathrm{k}Q/R^2$, attained just outside the sphere. Unconvinced of the validity of the existing justifications for the presence of this 'half-factor', all based upon Gauss's law, including a recent one by Ganci [17], I have developed here a rigorous mathematical proof that the half-factor actually takes place. Since our proof involves only basic calculus tools, it could well be included in introductory physics classes and textbooks.

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Suggested Reading

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