# On Some Results of Alan Baker

*Yann Bugeaud*

**Alan Baker, who died on the 4th of February of this year, was born in England on the 19th of August 1939. In 1965 he defended his doctoral dissertation titled 'Some Aspects of Diophantine Approximation' at Trinity College Cambridge under the guidance of Harold Davenport. It is a very unusual fact that eight of his papers, including [1], which is discussed in the next section, had appeared in print before he submitted his doctoral dissertation. Baker was awarded the Fields Medal in 1970 at the International Congress of Mathematicians at Nice. Other honors he received are detailed in the Article-in-a-Box in this issue.**

**In the present text, we will discuss his paper [1] and the 'Baker theory of linear forms in logarithms', which started with the series of four papers [2, 3, 4, 5] published in the British journal *Mathematika*. This new theory was an impressive breakthrough in the field of Diophantine approximation and we will briefly present some of its applications. For precise references to original papers, including those mentioned in the next sections, the reader is directed to the monographs [6, 7, 8, 9].**

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## Rational Approximation of Algebraic Numbers

Let $\xi$ be an irrational real number, which to simplify the exposition we assume to be in the interval $(1, 10)$. Let

$$\xi = a_0 + \sum_{j\geq 1} \frac{a_j}{10^j} = a_0 \cdot a_1 a_2 a_3 \ldots$$

denote its decimal expansion, where $a_j$ is in $\{0, 1, \ldots, 9\}$ for $j \geq 0$.

For every integer $J \geq 1$, the rational number $\xi_J := a_0{\cdot}a_1a_2a_3 \ldots a_J$ constructed by truncating the decimal expansion of $\xi$ is a rather

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A rational number $p/q$ is a good approximation to $\xi$ when the distance $|\xi - p/q|$ is small in comparison with the 'complexity' of $p/q$. Natural notions of complexity include the number of digits of $p$ plus that of $q$, or the maximum of $|p|$ and $|q|$.

good rational approximation to $\xi$. Namely, setting $p_J := 10^J \xi_J$, we get

$$|\xi - \xi_J| = \left|\xi - \frac{p_J}{10^J}\right| < \frac{1}{10^J}.$$

A rational number $p/q$ is a good approximation to $\xi$ when the distance $|\xi - p/q|$ is small in comparison with the 'complexity' of $p/q$. Natural notions of complexity include the number of digits of $p$ plus that of $q$, or the maximum of $|p|$ and $|q|$. Intuitively, if we can choose in an optimal way the $k$ digits of $p$ and the $k$ digits of $q$, then we may hope that the first $2k$ decimals of $p/q$ coincide with those of $\xi$. In the above naïve discussion, the digits of the denominators are imposed (the denominators are integer powers of 10), so we could choose only the digits of the numerators.

Let us now explain how one can do better, that is, how one can construct an infinite sequence of rational numbers $p/q$ such that $|\xi - p/q| < 1/q^2$.

Let $Q \geq 2$ be an integer. Place the fractional parts

$$\{\xi\}, \{2\xi\}, \ldots, \{(Q-1)\xi\}$$

in the interval $[0, 1]$. We thus have $Q - 1$ points in $(0, 1)$ which divide the interval $[0, 1]$ into $Q$ subintervals, the sum of their lengths being obviously equal to 1. Consequently, (at least) one of these subintervals has its length at most equal to $1/Q$. Since its endpoints can be written $a\xi + b$ and $a'\xi + b'$, with $a, a'$ in $\{0, 1, \ldots, Q-1\}$ and $b, b'$ in $\mathbf{Z}$, we get

$$0 < |(a\xi + b) - (a'\xi + b')| \leq \frac{1}{Q}.$$

Setting $q = |a - a'|$, this shows that there exists $p$ in $\mathbf{Z}$ such that

$$0 < |q\xi - p| \leq \frac{1}{Q},$$

and, since $1 \leq q \leq Q - 1$, we conclude that

$$\left|\xi - \frac{p}{q}\right| \leq \frac{1}{qQ} < \frac{1}{q^2}.$$

Since $\xi$ is irrational, by starting with arbitrarily large values of $Q$, we get infinitely many rational numbers $p/q$ such that $|\xi - p/q| <$

$1/q^2$. However, it is important to emphasize that the proof gives absolutely no information on the digits of $p$ and $q$.

A real number is effectively computable if one can write explicitly a suitable value for it.

We formalize what we have just discussed by introducing the irrationality exponent of an irrational number $\xi$, which measures the quality of approximation to $\xi$ by rational numbers.

In the sequel, we say that a real number is effectively computable if one can write explicitly a suitable value for it. Let us illustrate this notion with the following example. Let $S$ be a set of positive integers with the property that its largest element has no more than twice the number of decimal digits of its smallest element. It is easy to deduce that $S$ is a finite set. Said differently, there exists $C$ such that $S$ has at most $C$ elements. However, under our assumptions, it is not possible to find a suitable value for $C$. Thus, $C$ is not effectively computable (we say also that $C$ is ineffective). Things change completely if we know, for example, that $10^6$ is in $S$, in which case one can easily deduce that $S$ has no more than $10^{14}$ elements.

**Definition 2.1**. Let $\xi$ be an irrational real number. The real number $\mu$ is an irrationality measure for $\xi$ if there exists a positive real number $C(\xi, \mu)$ such that every rational number $\frac{p}{q}$ with $q \geq 1$ satisfies

$$\left|\xi - \frac{p}{q}\right| > \frac{C(\xi, \mu)}{q^{\mu}}.$$

If, moreover, the real number $C(\xi, \mu)$ is effectively computable, then $\mu$ is an effective irrationality measure for $\xi$. We denote by $\mu(\xi)$ (*resp.* $\mu_{\text{eff}}(\xi)$) the infimum of the irrationality measures (*resp.* effective irrationality measures) for $\xi$ and call it the irrationality exponent (*resp.* effective irrationality exponent) of $\xi$.

Clearly, $\mu_{\text{eff}}(\xi)$ is always larger than or equal to $\mu(\xi)$. We have shown that every irrational real number $\xi$ satisfies $\mu(\xi) \geq 2$ and we claim that equality holds for almost every $\xi$ (with respect to the Lebesgue measure). This follows from an easy covering argument using the fact that for every positive $\varepsilon$, the series $\sum_{q \geq 1} q^{-1-\varepsilon}$ converges. Furthermore, it is believed that $\mu(\pi) = \mu(\log 2) = 2$, but we are very far away from having a proof.

A complex number is called 'algebraic' if it is a root of a non-zero polynomial with integer coefficients and a complex number which is not algebraic is called 'transcendental'.

Recall that a complex number is called 'algebraic' if it is a root of a non-zero polynomial with integer coefficients and a complex number which is not algebraic is called 'transcendental'. The degree of an algebraic number $\theta$ is the smallest positive integer $d$ such that $\theta$ is a root of an integer polynomial of degree $d$. In 1844, Liouville established that the irrationality exponent of any irrational algebraic number is at most equal to its degree. In fact, his result being completely explicit, it addresses the effective irrationality exponent.

**Theorem 2.2 (Liouville, 1844)**. *Let $\theta$ be an irrational algebraic number of degree $d$. There exists a positive real number $C(\theta)$, which can be given explicitly, such that the inequality*

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C(\theta)}{q^d}$$

*holds for all rational numbers $\frac{p}{q}$ with $q \geq 1$. Consequently, we have*

$$\mu_{\text{eff}}(\theta) \leq d.$$

Roth's celebrated theorem, established in 1955, asserts that from the point of view of rational approximation, an irrational real algebraic number behaves like almost every real number.

**Theorem 2.3 (Roth, 1955)**. *Every irrational real algebraic number $\xi$ satisfies $\mu(\xi) = 2$.*

Roth's theorem considerably improves Liouville's theorem, but at the cost of effectivity. It says nothing on the effective irrationality exponent of an irrational algebraic number and it was (and it is still!) a big challenge to get effective improvements of Liouville's theorem. In the course of his PhD thesis, Baker obtained such effective improvements for some special algebraic numbers, including $\sqrt[3]{2}$.

**Theorem 2.4 (Baker, 1964)**. *We have $\mu_{\text{eff}}(\sqrt[3]{2}) \leq 2.955$.*

The proof of Theorem 2.4 depends on identities involving hypergeometric functions, inspired by previous works of Thue and Siegel. A key ingredient is the equality $5^3 - 2 \cdot 4^3 = -3$, showing that $\sqrt[3]{2}$ is close to $5/4$.

Let us explain the connection between rational approximation to $\sqrt[3]{2}$ and the Diophantine equation $X^3 - 2Y^3 = m$, where $m$ is a fixed positive integer. Assume that we can prove the existence of positive real numbers $C, \tau$ such that

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| \geq \frac{C}{q^{3-\tau}}, \quad \text{for all integers } p, q \text{ with } q \geq 1. \qquad (2.1)$$

Let $m$ be a positive integer and $x, y$ be positive integers (we restrict our attention to positive integers, only to get some minor simplification in the discussion below) such that

$$x^3 - 2y^3 = m. \qquad (2.2)$$

Clearly, we have $x > y$ and we can assume that $x < 2y$ (since, otherwise, $x \leq 6m$). Let $\mathrm{j} = \mathrm{e}^{2\mathrm{i}\pi/3}$ be a primitive cube root of unity. Observe that $|x - \mathrm{j}\sqrt[3]{2}y| \geq x$. Since

$$(x - \sqrt[3]{2}y)\,(x - \mathrm{j}\sqrt[3]{2}y)\,(x - \mathrm{j}^2\sqrt[3]{2}) = m,$$

we get

$$|x - \sqrt[3]{2}y| \leq \frac{m}{x^2} \leq \frac{m}{y^2}.$$

By dividing by $y$ and combining with (2.1), this shows that

$$x \leq (m/C)^{1/\tau}.$$

This shows that if $C$ and $\tau$ can be explicitly given, then we get an explicit upper bound for the solutions to (2.2). Equation (2.2) is an example of a Thue equation.

Let $F(X, Y)$ be an homogeneous, binary, integer polynomial of degree at least three and such that $F(X, 1)$ has at least three distinct roots. Let $m$ be a non-zero integer. In 1909, Axel Thue established that the equation

$$F(x, y) = m, \quad \text{in integers } x, y, \qquad (2.3)$$

has only finitely many solutions. This equation is called the Thue equation. Thue's method, however, is ineffective, in the sense that it does not yield upper bounds for the absolute values of the solutions of (2.3). The first general effective result on Thue's equations was established by Baker. It was subsequently refined by

Baker and by Feldman, who established the following effective statement.

**Theorem 2.5 (Feldman, 1971)**. *Let $F(X, Y)$ be an homogeneous, irreducible polynomial with integer coefficients and of degree at least three. Let m be a non-zero integer. Then, there exists an effectively computable positive real number C, depending only on $F(X, Y)$, such that all the integer solutions $x, y$ to the Diophantine equation*

$$F(x, y) = m$$

*satisfy* $\max\{|x|, |y|\} \leq |2m|^C$.

A straightforward consequence of Theorem 2.5 (to see the connection between Theorems 2.5 and 2.6, argue as in the particular case of $X^3 - 2Y^3 = m$ discussed above) is the following effective improvement of Liouville's theorem.

**Theorem 2.6**. *Let $\theta$ be a real algebraic number of degree d with $d \geq 3$. Then, there exists an effectively computable positive real number $\tau$, depending only on $\theta$, such that $\mu_{\text{eff}}(\theta) \leq d - \tau$.*

It should be noted that the proof of Theorem 2.6 yields a suitable value for $\tau$ which is very small. In the special case of $\sqrt[3]{2}$, Theorem 2.4 gives a much stronger result.

The key ingredient for the proofs of Theorems 2.5 and 2.6 is the theory of linear forms in logarithms of algebraic numbers, developed by Baker from 1966 and discussed in the next section.

## Linear Forms in Logarithms of Algebraic Numbers

Leonhard Euler established the relationship $e^{i\pi} = -1$ and gave explicitly the continued fraction expansions of e and $e^2$. He also made a conjecture concerning the nature of quotients of logarithms of rational numbers.

This section is mainly borrowed from [7].

In 1748 Leonhard Euler published *Introductio in analysin infinitorum* where, among several fundamental results, he established the relationship $e^{i\pi} = -1$ and gave explicitly the continued fraction expansions of e and $e^2$. He also made a conjecture concerning the nature of quotients of logarithms of rational numbers, which can be formulated as follows:

*For any two positive rational numbers r, s with r different from* 1, *the number* $\frac{\log s}{\log r}$ *is either rational (in which case there are non-zero integers* $a, b$ *such that* $r^a = s^b$*) or transcendental.*

Euler's conjecture implies, for example, that $2^{\sqrt{2}}$ is irrational (if it were rational, then $\log 2^{\sqrt{2}}$ divided by $\log 2$, which is equal to $\sqrt{2}$, would be rational or transcendental). It can be reformulated as follows:

*If* $a$ *is a positive rational number different from* 1 *and* $\beta$ *an irrational real algebraic number, then* $a^\beta$ *is irrational.*

If $a$ is a positive rational number different from 1 and $\beta$ an irrational real algebraic number, then $a^\beta$ is irrational.

In 1900, David Hilbert proposed a list of twenty-three open problems and presented ten of them in Paris at the second International Congress of Mathematicians. His seventh problem expands the arithmetical nature of the numbers under consideration in Euler's conjecture and asks whether (observe that $e^\pi = (-1)^{-i}$)

*The expression* $\alpha^\beta$ *for an algebraic base* $\alpha \neq 0, 1$ *and an irrational algebraic exponent* $\beta$*, e.g. the number* $2^{\sqrt{2}}$ *or* $e^\pi$*, always represents a transcendental or at least an irrational number.*

Here and below, unless otherwise specified, by algebraic number we mean complex algebraic number. Hilbert believed that the Riemann Hypothesis would be settled long before his seventh problem. This was not the case: the seventh problem was eventually solved in 1934, independently and simultaneously, by Aleksandr Gelfond and Theodor Schneider, by different methods. They established that for any non-zero algebraic numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\log \alpha_1$ and $\log \alpha_2$ linearly independent over the rationals (here and below, log denotes the principal determination of the logarithm function), we have

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0.$$

Since the formulation is different, let us add some explanation. Under the hypotheses of Hilbert's seventh problem, the complex numbers $\log \alpha$ and $\log \alpha^\beta$ are linearly independent over the rationals and, assuming furthermore that $\alpha^\beta$ is algebraic, we derive from the Gelfond–Schneider theorem that $\beta$, equal to the quotient of the logarithm of $\alpha^\beta$ by the logarithm of $\alpha$, cannot be algebraic, a contradiction.

Subsequently, Gelfond derived an effective lower bound for $|\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2|$, and a few years later, he realized that an extension of his result to linear forms in an arbitrarily large number of logarithms of algebraic numbers would enable one to solve many challenging problems in Diophantine approximation and in the theory of Diophantine equations.

This program was realized by Alan Baker in a series of four papers [2, 3, 4, 5] published between 1966 and 1968.

**Theorem 3.1**. *Let n be a positive integer. Let* $\alpha_1, \dots, \alpha_n$ *be non-zero algebraic numbers and* $\log \alpha_1, \dots, \log \alpha_n$ *any determinations of their logarithms. If* $\log \alpha_1, \dots, \log \alpha_n$ *are linearly independent over the rationals, then* $1, \log \alpha_1, \dots, \log \alpha_n$ *are linearly independent over the field of algebraic numbers.*

It readily follows from Theorem 3.1 that the complex number $e^{\beta_0}\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$ is transcendental for all non-zero algebraic numbers $\alpha_1, \dots, \alpha_n, \beta_0, \dots, \beta_n$. Furthermore, any expression of the form

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

where $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are non-zero algebraic numbers and $\beta_0$ is algebraic, vanishes only in trivial cases. A natural question is then to bound from below its absolute value (when non-zero).

For the sake of simplification, we assume in the discussion below that the algebraic numbers involved are all rational numbers. Let $n \geq 2$ be an integer. For $j = 1, \dots, n$, let $\frac{x_j}{y_j}$ be a non-zero rational number, $b_j$ a non-zero integer, and set

$$B := \max\{3, |b_1|, \dots, |b_n|\} \quad \text{and} \quad A_j := \max\{3, |x_j|, |y_j|\}. \quad (3.1)$$

We consider the rational number

$$\Lambda := \Big(\frac{x_1}{y_1}\Big)^{b_1} \cdots \Big(\frac{x_n}{y_n}\Big)^{b_n} - 1. \quad (3.2)$$

Many classical Diophantine equations can be reduced to expressions of the form (3.2), where the rational numbers $x_i/y_i$ may be replaced by algebraic numbers.

Recall that $|x|/2 \leq |\log(1 + x)| \leq 2|x|$ holds for every real number $x$ with $|x| \leq 1/2$. Since we wish to bound $|\Lambda|$ from below, we may assume that $|\Lambda| \leq \frac{1}{2}$. Then, the *linear form in logarithms of rational numbers* $\Omega$, defined by

$$\Omega := \log(1 + \Lambda) = b_1 \log \frac{x_1}{y_1} + \cdots + b_n \log \frac{x_n}{y_n},$$

satisfies

$$\frac{|\Lambda|}{2} \leq |\Omega| \leq 2|\Lambda|.$$

A trivial estimate of the denominator of (3.2) gives that $\Lambda = 0$ or

$$\log|\Lambda| \geq -\sum_{j=1}^{n} |b_j| \log \max\{|x_j|, |y_j|\} \geq -B \sum_{j=1}^{n} \log A_j. \tag{3.3}$$

The dependence on the $A_j$'s in (3.3) is very satisfactory, unlike the dependence on $B$. For applications to Diophantine problems, we require a better estimate in terms of $B$ than the one given in (3.3), even if it comes with a weaker one in terms of the $A_j$'s. For example, replacing $B$ by $o(B)$ is sufficient for many applications, but not for all.

At the end of the 80s, it was established that with $\Lambda, A_1, \ldots, A_n$, and $B$ as in (3.1) and (3.2), there exists an effectively computable real number $c(n)$, depending only on the number $n$ of rational numbers involved in (3.2), such that the lower estimate

$$\log\left|\left(\frac{x_1}{y_1}\right)^{b_1} \cdots \left(\frac{x_n}{y_n}\right)^{b_n} - 1\right| \geq -c(n) \log A_1 \cdots \log A_n \log B$$

holds, when $\Lambda$ is non-zero. Since then, several authors managed to considerably reduce the value of the real number $c(n)$. However, it remains an open problem to replace the product $\log A_1 \cdots \log A_n$ by the sum $\log A_1 + \cdots + \log A_n$.

Parallel to the development of the theory of linear forms in complex logarithms, progress has been regularly made towards its $p$-adic analogue, where $p$ is a prime number. This allows us to get rather good effective upper bounds for the greatest integral power of $p$ which divides an expression like (3.2).

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## Some Applications to Diophantine Problems

The applications of Baker's theory are not limited to Diophantine equations and Diophantine problems, and some of them also address the theory of dynamical systems.

It is not our purpose to present a list of applications of Baker's theory. Let us, however, emphasize that applications are not limited to Diophantine equations and Diophantine problems, and some of them also address the theory of dynamical systems.

It is not at all obvious that many classical Diophantine equations, including Thue equations, can be reduced to expressions like (3.2), with algebraic numbers in place of rational numbers. Results from algebraic number theory are heavily needed. At the end of the 60s, Baker showed how to derive from estimates of linear forms in logarithms explicit upper bounds for the integer solutions $x, y$ of equations of the form $f(x) = y^n$, where $f(X)$ is an integer polynomial and $n$ a fixed integer (under some necessary assumptions, of course; recall that the Pell equation $X^2 - 2Y^2 = 1$ has infinitely many solutions). By earlier works of Siegel, these equations were already known to have only finitely many solutions, but Siegel's result is ineffective.

Apart from the aspect of effectivity, the theory of linear forms in logarithms appears to be much more powerful than the methods developed by Thue and Siegel. Indeed, it also applies to certain families of exponential Diophantine equations (this terminology means that one or several exponents are unknown). As first proved by Schinzel and Tijdeman in 1976, Baker's theory allows one to get explicit upper bounds for the integer solutions $x, y, z$ of equations of the form $f(x) = y^z$, where $z \geq 2$ and $f(X)$ is an integer polynomial of degree at least 3. Such a result was at that time completely new. A spectacular achievement was the proof by Robert Tijdeman [10] in 1976 that the Catalan equation $x^m - y^n = 1$, in the integer unknowns $x, y, m, n$ all greater than 1, has only finitely many solutions.

**Theorem 4.1 (Tijdeman, 1976).** *There exists an effectively computable number C such that all the integers $x, y, m, n$ with*

$$x^m - y^n = 1, \quad \min\{x, y, m, n\} \geq 2, \tag{4.1}$$

*satisfy* $\max\{x, y, m, n\} \leq C$.

Preda Mihăilescu [11] established in 2002 that $3^2 - 2^3 = 1$ is the only solution to (4.1). His original proof appeals, at one of its many steps, to Baker's theory. Two years later he managed to find a different argument for this step, thus obtaining a proof of his theorem completely independent of Baker's theory.

In the 1970s and the 1980s, many equations or families of equations were proved to have only finitely many solutions by means of Baker's method. This area was at that time flourishing and developing very rapidly, both from a theoretical point of view and regarding its applications. The bounds for the solutions were however very large and we were unable to solve completely the equations in consideration. Since about 30 years, numerous Diophantine equations have been completely solved, which seemed previously to be far beyond our possibilities. There are two main explanations. The first one is a theoretical improvement: the size of the numerical constants appearing in the estimates for linear forms in logarithms has been substantially reduced and is now (at least in the case of two logarithms) rather satisfactory. The second one is the spectacular development of the algorithmic number theory.

For instance, we have now at our disposal efficient algorithms which enable us to solve any Thue equation of small degree, say of degree less than thirteen, and with small coefficients. Further, there are examples of Thue equations of high degree which are completely solved. The following one is a result of Hanrot.

**Theorem 4.2**. *The Thue equations*

$$\prod_{1\le k\le 2000}\left(X - 2\cos\left(\frac{2\pi k}{4001}\right)Y\right) = \pm 1, \pm 4001 \qquad (4.2)$$

*have no non-trivial integral solutions.* We should however point out that this spectacular result absolutely does not mean that we are now able to solve any Thue equation of degree less than two thousand! Equation (4.2) has a very particular shape: the right-hand side is indeed a cyclotomic polynomial.

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Bennett managed to solve completely a two-parametric family of Thue equations.

**Theorem 4.3**. *Let $b \geq 1$ and $n \geq 3$ be integers. Then, the Diophantine equation*

$$(b + 1)x^n - by^n = 1$$

*has no solution in positive integers $x$ and $y$ with $x \geq 2$.*

Shorey and Tijdeman proved in an effective way that only finitely many integers greater than 2 of the form $11\ldots11$, i.e. with only the digit 1 in their decimal representation, can be perfect powers (a perfect power is an integer of the form $a^b$, with $a, b$ positive integers and $b \geq 2$).

In 1976, Shorey and Tijdeman proved in an effective way that only finitely many integers greater than 2 of the form $11\ldots11$, i.e. with only the digit 1 in their decimal representation, can be perfect powers (a perfect power is an integer of the form $a^b$, with $a, b$ positive integers and $b \geq 2$). This was the first step towards a proof of a longstanding conjecture claiming that none of these numbers is a pure power, which has recently been settled by Bugeaud and Mignotte.

**Theorem 4.4**. *With the exception of* 1*, no integer with only the digit 1 in base ten can be a pure power.*

The original proof used sharp estimates for linear forms in two 2-adic logarithms, as well of a great amount of computer calculations. Some of them could be avoided by application of Theorem 4.3. To see this, let us observe that Theorem 4.4 amounts to solve the Diophantine equation $(10^n - 1)/(10 - 1) = y^q$. The existence, for every $q \geq 2$, of the solution $n = 1, y = 1$ makes this problem difficult. One step of the proof is to show that there is no solution with $n$ congruent to 1 modulo $q$. Assume that $n = \nu q + 1$, with $\nu$ a positive integer. Then, the equation becomes

$$10 \cdot (10^\nu)^q - 9 \cdot y^q = 1,$$

which, by Theorem 4.2, has no solutions with $\nu \geq 1$.

We further display the statement of a result of Bugeaud, Mignotte and Siksek, whose proof depends greatly on Baker's theory and, also, on the modularity theorem. Recall that the Fibonacci sequence $(F_n)_{n \geq 0}$ is defined by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$.

**Theorem 4.5**. *The only perfect powers in the Fibonacci sequence are* 0, 1, 8, *and* 144.

On a completely different note, we mention a result of Baker and Coates, who used a linear form in two 2-adic logarithms to make effective a theorem of Mahler on the distribution of the fractional parts of integer powers of 3/2. Denoting by $\| \cdot \|$ the distance to the nearest integer, Mahler proved in 1957 that for every positive $\varepsilon$, there exists an integer $n_0(\varepsilon)$, depending only on $\varepsilon$, such that $\|(3/2)^n\| > 2^{-\varepsilon n}$, for every $n > n_0(\varepsilon)$. Since his argument uses Ridout's generalization of Roth's theorem, the integer $n_0(\varepsilon)$ is not effectively computable.

Baker's theory allows one to get lower bounds for the greatest prime factor of integer values of integer polynomials.

**Theorem 4.6**. *There exists a positive $\delta$ such that*

$$\Big\|\Big(\frac{3}{2}\Big)^n\Big\| \geq \frac{1}{2^{1+(1-\delta)n}}, \quad \textit{for every } n \geq 1.$$

Baker's theory allows one to get lower bounds for the greatest prime factor of integer values of integer polynomials. For an integer $n \geq 2$, we denote by $P[n]$ its greatest prime divisor.

**Theorem 4.7**. *There exists a positive, effectively computable real number c such that*

$$P[n^2 - 1] \geq c \, \frac{\log\log n}{\log\log\log\log n} \, \log\log\log n, \quad \textit{for } n \geq 16.$$

We conclude with a very recent result. In 1965 Erdős conjectured that

$$\lim_{n \to +\infty} \frac{P[2^n - 1]}{n} = +\infty.$$

With the help of the theory of linear forms in logarithms, Stewart was able to solve this conjecture in 2013.

**Theorem 4.8**. *There exists an effectively computable integer $n_0$ such that for any integer n greater than $n_0$, we have*

$$P[2^n - 1] > n \, \exp\Big(\frac{\log n}{104 \log\log n}\Big).$$

The proof of Theorem 4.8 uses properties of the integers of the form $2^n - 1$ and, in a very clever way, estimates for linear forms in $p$-adic logarithms.

For additional applications, the reader is directed to the monographs [7, 8] and, of course, to the original articles.

## Suggested Reading


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