Alan Baker

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Alan Baker studied at Stratford Grammar School and entered University College, London where he was awarded his B.Sc. degree with First-Class Honours in 1961. After receiving an M.A. degree from Trinity College, Cambridge, Baker continued his research and received his doctoral degree in 1965; his advisor was Harold Davenport. Remarkably, eight of his papers were published even before he submitted his doctoral dissertation. Baker's doctoral dissertation was titled *Some Aspects of Diophantine Approximation*. From 1974 until 2006, Baker was a Professor of pure mathematics at Cambridge University, after which he was made an Emeritus and remained so until his passing away.

Baker's doctoral advisor Harold Davenport once commented, "I had two very good students: Baker, to whom I would give a problem and he would return with a very good solution, and Conway, to whom I would give a problem and he would return with a very good solution to another problem."

Baker was awarded the Fields Medal in 1970 at the Nice ICM at the age of 31. His Fields Medal work was described thus by Paul Turan:

"Baker's achievement is all the more impressive given David Hilbert's prediction that the Riemann hypothesis, which remains unproven, would be settled long before the proof of the transcendence of α^β....... I remark that his work exemplifies two things very convincingly. Firstly, that beside the worthy tendency to start a theory in order to solve a problem it pays also to attack specific difficult problems directly. Particularly is this the case with such problems where rather singular circumstances do not make it probable that a solution would fall out as an easy consequence of a general theory. Secondly, it shows that a direct solution of a deep problem develops itself quite naturally into a healthy theory and gets into early and fruitful contact with other significant problems of mathematics. So, let the two different ways of doing mathematics live in peaceful coexistence for the benefit of our science."

Baker's seminal work is in the area of transcendental number theory. Astoundingly, the Russian–German duo of A O Gelfond and T Schneider gave an affirmative solution to Hilbert's seventh problem as early as in 1934. They proved that the logarithm of an algebraic number to an algebraic base is either rational or transcendental (the latter is a complex number that is not the root of a non-zero polynomial with rational coefficients). In particular, $e^\pi = (-1)^{-i}$ is transcendental.

Gelfond–Schneider's techniques had been stretched to their limit when Baker started working on these problems on his own. Baker developed what is now known as the 'theory of linear forms in logarithms'. In particular, his theorem implies the transcendence of:

$$a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n},$$

where $1, b_1, \cdots, b_n$ are algebraic numbers which are linearly independent over $\mathbf{Q}$ and a_i's are algebraic and different from 0 and 1. Baker's theory yields the solutions to a large number of problems in number theory. This includes Gauss' famous class number problem for which Gelfond–Schneider's method did not apply; Baker's new approach spectacularly solved this. Using his theory of linear forms in logarithms, Baker was able to effectively find all the integer solutions of the Mordell Diophantine equations $y^2 = x^3 + k$. In fact, Baker showed that for a Diophantine equation $f(x, y) = m$, where m is a positive integer, and $f(x, y)$ is an irreducible form with integer coefficients and of degree at least 3, there is an effective bound B that only depends on the degree and the coefficients of the function, so that $max(|x_0|, |y_0|) \leq B$, for any solution (x_0, y_0). Baker's ideas were developed by others subsequently. In 1983, Faltings proved the full Mordell conjecture asserting finiteness of the number of solutions of Diophantine equations of this type which includes the Fermat equation; but the solutions cannot be computed explicitly. Baker's insights have also found applications in fields of mathematics remote from number theory.

Baker wrote two textbooks on number theory (a concise one and a comprehensive one), both regarded as masterpieces. He wrote a number of other books including a monograph called *Transcendental Number Theory* in 1975 and a text named *Logarithmic Forms and Diophantine Geometry* with G Wüstholz in 2007.

H Halberstam writes in a review of *Transcendental Number Theory* that: "Within the space of a mere 130 pages the author gives a panoramic account of modern transcendence theory, based on his own Adams Prize essay. The fact that this is now 'a fertile and extensive theory, enriching wide-spread branches of mathematics' is due in large measure to the author himself, who was awarded in 1970 a Fields Medal for his contributions. The prose is clear and economical yet interspersed with flashes of colour that convey a sense of personality; and each chapter begins with a helpful summary of the subsequent matter. The mathematical argument at all stages is highly condensed, as, indeed, is inevitable in a short research monograph covering so much ground. One might reproach the author for not having been more merciful to the beginner; but even a beginner can gain from the book a clear impression of what are the major achievements to date in this profoundly difficult field and which are the outstanding problems, while for others there is here a wealth of material for numerous fruitful study-groups."

An interesting comment made by Baker on the history of transcendental number theory asserts:

"Well, what does this tell us about the historical evolution of mathematics? First it is clear that a very important role has been played by a few key problems, centres of attraction, in Professor Dieudonné's terminology. This may be more true of number theory than other branches of mathematics but I believe that all good work has been guided to some extent by such centres. The general trend of the particular field that I have been discussing is difficult to summarise, since it has involved in its development many novel twists and turns; but one obvious element in the evolution has been the successful blending, or fusion, of ideas from number theory and algebra with the progressively wider use of classical function theory. And it is this convergence of diverse concepts that forms the essential ingredient, I believe, in the creation of an active theory. According to Professor Dieudonné, the study of transcendental numbers is only just on its way to becoming a 'method'. Given, however, the diverse nature of the problems which it has been instrumental in solving, there seems little doubt that it reached the latter stage several years ago, and it would appear, in fact that it is already on the path of becoming, in Professor Dieudonné's language, a centre of radiation."

Baker never married; during his speech in the 1999 International Conference in Zurich to celebrate his 60th birthday, he expressed regret at this. His hobbies included travelling around the world, as well as photography and theatre. Apart from the Fields Medal, Baker received many other honours such as the Adams prize, election to the Royal Society and the Academia Europaea, and election to the honorary fellowship of UCL.

We end with a description of a gem of a paper in 2004 by Baker on the abc–conjecture that elucidates ideas he had outlined earlier in 1996. Motivated by a conjecture by Szpiro on elliptic curves connecting its conductor and its discriminant, Oesterlé made a conjecture on integers which was modified by Masser to a form now known as the abc-conjecture. It asserts:
If a, b, c are coprime integers satisfying $a + b + c = 0$, then for any $\epsilon > 0$,

$$max(|a|, |b|, |c|) \leq c_\epsilon N^{1+\epsilon},$$

where the 'conductor' N is the product of the distinct prime factors of abc and c_ϵ is a constant depending only on ϵ.
This conjecture is known to be one of the key open problems of mathematics. It implies for instance that the generalized Fermat equation $ax^k + by^l + cz^m = 0$ has only finitely many solutions in coprime integers x, y, z and exponents k, l, m when $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$. It also implies Faltings's theorem on the Mordell conjecture and the famous results of Thue–Siegel–Roth. In fact, a uniform version of abc–conjecture over number fields implies that the Dirichlet L-functions do not have a Siegel zero (a possible zero not in the critical strip). In the 2004 paper, Baker puts forth an explicit version of the abc–conjecture based on the theory of linear forms

in logarithms. Baker conjectured:

$$max(|a|, |b|, |c|) \leq \frac{6}{5}\frac{N\log(N)^{\omega}}{\omega!},$$

where ω is the number of distinct primes dividing abc and N is the conductor of abc. This explicit version implies Fermat's last theorem for exponent at least 7 immediately. We mention just one more simple-to-state finiteness implication due to Shorey and Laishram of the explicit Baker conjecture:

If $a, b, n > 1$ and the digits of a^n in base b are all equal to 1 (with the number of digits > 2), then the only solutions are

$$11^2 = 1 + 3 + 3^2 + 3^3 + 3^4, \quad 20^2 = 1 + 7 + 7^2 + 7^3, \quad 7^3 = 1 + 18 + 18^2.$$

We end with:

Baker meditated on the transcendental
and proved results monumental.
e, π, and e^{π} we knew,
are not algebraic, but there are quite a few
others like $\sum \beta_i \log \alpha_i$ as well!

Alan baker – bit by bit –
made abc quite explicit.
Notorious conjectures difficult to bust
were now made to bite the dust.
Kudos to this celebrated Brit!

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