In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

**A Geometrical Visualisation of**

\[ \frac{x}{1+x} < \ln(1+x) < x - \frac{x^2}{2(1+x)} \text{ for } x > 0 \]

In this classroom article, we present a geometrical interpretation of a well-known inequality \[ \frac{x}{1+x} < \ln(1+x) < x - \frac{x^2}{2(1+x)} \text{ for } x > 0. \]

1. **Geometrical visualisation of** \[ \frac{x}{1+x} < \ln(1+x) < x - \frac{x^2}{2(1+x)} \text{ for } x > 0 \]

From the definition of the definite integral, we have (see *Figure 1*):

\[
\text{Area of the region bounded by the curve } y = \frac{1}{x} \\
\text{and the ordinates } y = 1 \text{ and } y = x + 1 \\
= \int_1^{x+1} \frac{1}{x} \, dx \\
= \ln(x + 1).
\]

**Keywords**

Inequality, logarithm, visual proof.
Figure 1. Geometrical visualisation of $\frac{1}{1+x} < \ln(1 + x) < x - \frac{x^2}{2(1+x)}$ for $x > 0$.

Again,

Area of the trapezium $ACQP$

\[
\frac{1}{2}PQ \times (AP + CQ) = \frac{1}{2}x \left(1 + \frac{1}{1 + x}\right) = x - \frac{x^2}{2(1 + x)}.
\]

Also,

Area of the rectangle $PQCD = PQ \times CQ = x \times \frac{1}{1 + x}$.

From the figure, it is clear that:

Area of the rectangle $PQCD$

< Area of the region bounded by the curve $y = \frac{1}{x}$

and the ordinates $y = 1$ and $y = x + 1$.

< Area of the trapezium $ACQP$.

That is, $\frac{x}{1+x} < \ln(1 + x) < x - \frac{x^2}{2(1+x)}$.

Remark 1.1. Since

\[
x - \frac{x^2}{2(1 + x)} = \frac{x}{1 + x} \left(1 + \frac{x}{2}\right).
\]
we can also express the main inequality in the following equivalent forms:

\[
\frac{x}{1+x} < \ln(1+x) < \frac{x}{1+x} \left(1 + \frac{x}{2}\right) \quad \text{for all } x > 0; \quad (1)
\]

\[
x < (1 + x) \ln(1 + x) < x + \frac{x^2}{2} \quad \text{for all } x > 0; \quad (2)
\]

\[
1 < (1 + \frac{1}{x}) \ln(1 + x) < 1 + \frac{x}{2} \quad \text{for all } x > 0. \quad (3)
\]

**Remark 1.2.** Here we shall give an idea of the closeness of the expression of inequality (2) using power series. We know that for \(0 < x < 1\),

\[
\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad \text{and}
\]

\[
(1+x)\ln(1+x) = x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^6}{30} - \ldots.
\]

Now,

\[
(1+x)\ln(1+x) = x + \left(\frac{x^2}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{12} - \frac{x^5}{20}\right) + \ldots
\]

\[
> x + \left(\frac{x^2}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{12} - \frac{x^5}{20}\right) + \ldots
\]

\[
> x.
\]

Again,

\[
(1+x)\ln(1+x) = x + \frac{x^2}{2} - \left(\frac{x^3}{6} - \frac{x^4}{12}\right) - \left(\frac{x^5}{20} - \frac{x^6}{30}\right) - \ldots
\]

\[
< x + \frac{x^2}{2} - \left(\frac{x^3}{6} - \frac{x^4}{12}\right) - \left(\frac{x^5}{20} - \frac{x^6}{30}\right) - \ldots
\]

\[
< x + \frac{x^2}{2}.
\]

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Suggested Reading


