Maryam Mirzakhani

The Master Artist of Curved Surfaces

Nikita Agarwal, Riddhi Shah and Geetha Venkataraman

The article outlines the life and work of Maryam Mirzakhani – the first and the only female winner of the Fields Medal since its inception in 1936. Mirzakhani is an inspiration to young women around the world, to believe in their own abilities and pursue their academic dreams.

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1. Life-Sketch

As a young girl in Teheran, Maryam Mirzakhani's interest in mathematics was sparked off by her elder brother. He had related to her the story of how Gauss added the first 100 numbers using an ingenious bit of mathematical thinking. Growing up in Iran, Maryam has related the wonderful support she received from her parents, family of three siblings, and her high school teachers.

Her childhood was spent in Teheran during the Iran–Iraq war, which lasted from 1980 to 1988. In an interview given to the Clay Mathematical Institute [1] in 2008, Maryam said, "I was very lucky in many ways. The war ended when I finished elementary school; I couldn't have had the great opportunities that I had if I had been born ten years earlier."

After elementary school, Maryam sat for an exam to be admitted to the Farzanegan middle school for girls in Tehran. This school's aim was to nurture exceptional talent and educate the brightest of the students. As it happened, Maryam did not do well in mathematics in her first year at Farzanegan, and was told by her then teacher that she did not have mathematical talent. In fact, she thought then that she would be an author. She read a lot of books as there was a bookstore near her middle school.

A change of teacher the next year saw Maryam flourish math-

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Keywords

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ematically. She also made a lifelong friend at school in Roya Beheshti, who was also mathematically talented. The two girls had each other for company, and this did make a difference. The combination of a strong-willed principal at the Farzanegan high school, and two young girls who were hooked to mathematical problem solving, formed a formidable team. Soon Maryam and Roya broke into what was once considered a boys-only bastion, by being selected to represent Iran in the International Mathematical Olympiad (IMO). Maryam won a gold medal the first time she participated in 1994 with a score of 41 out of 42 and then again in 1995, this time with a perfect score.

In 1995, Maryam joined the Sharif University of Technology in Teheran. She completed her Bachelor of Science (BS) in 1999. By then she had already published three papers, of which two were in graph theory. She enjoyed problem solving and informal reading groups that she participated in with her peers at Sharif University. After completing BS from Teheran, she took admission at Harvard University for graduate work.

Curtis McMullen had been awarded the Fields Medal in 1998 and was appointed as Professor at Harvard the same year. Maryam attended his seminars at Harvard. Maryam's background before she came to Harvard was more in combinatorics and algebra. She was also familiar with complex analysis. Her initial perception of Curtis' lectures was that she did not understand much but could appreciate the way he brought alive topics and the way he could convey their simplicity and elegance. McMullen became Maryam's doctoral advisor.

To quote Maryam [1], "So I started asking him questions regularly, and thinking about problems that came out of these illuminating discussions. His encouragement was invaluable. Working with Curt had a great influence on me, though now I wish I had learned more from him! By the time I graduated, I had a long list of raw ideas that I wanted to explore."

Curtis, on the other hand, had the following to say about Maryam [2]: "She had a sort of daring imagination. She would formulate in her mind an imaginary picture of what must be going on, then come to my office and describe it. At the end, she would turn to me and say, “Is it right?" I was always very flattered that she thought I would know."

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Maryam worked on hyperbolic surfaces for her doctoral work. In 2004, she earned her doctorate from Harvard University for her 130-page thesis titled *Simple geodesics on hyperbolic surfaces and volume of the moduli space of curves* [3]. She was awarded the Leonard M and Eleanor B Blumenthal award for her thesis, which was judged as outstanding. After her doctoral degree, Maryam had a chance to continue at Harvard as a Junior Fellow but chose to take up an Assistant Professorship at Princeton University equipped with a Clay Research Fellowship.

Maryam felt that the opportunities offered by the Clay Fellowship were excellent and allowed her to work on problems at her own pace, which she proclaimed was slow. In three years, during the Fellowship, Maryam produced several profound papers on curved surfaces. These were published in leading journals in mathematics, like the *Annals of Mathematics*, *Inventiones Mathematicae*, and the *Journal of the American Mathematical Society*. After she completed the Fellowship in 2008, she joined Stanford University as a Professor of Mathematics.

While at Princeton, Maryam met Jan Vondrák, a Czech national who had a doctorate in computer science and also a PhD in applied mathematics from the Massachusetts Institute of Technology. Jan was a post-doctoral teaching fellow at Princeton. They married in 2005 and had a daughter Anahita in 2011.

Maryam was an invited speaker at the International Congress of Mathematicians (ICM) of 2010, held at Hyderabad, India. The first International Conference of Women Mathematicians (ICWM), immediately preceded the Congress, and Maryam was one of the eight women who delivered lectures in the event. Her talks were a treat, visually and mathematically, and left a lasting impression on the audience.

In 2006, during the Clay Fellowship, she began collaborating

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with a Chicago University mathematician, Alex Eskin. They co-authored several seminal works, including *Counting closed geodesics in moduli space* [4], published in 2011. Maryam continued to spear ahead with outstanding mathematical contributions that furthered our understanding of curved spaces. She has often been pictured with large sheets of paper spread on the floor on which she would visualise her mathematics of curved surfaces. Her daughter would remark that her mother was painting [5]. Maryam received innumerable awards and prizes for her mathematical work and was finally awarded, in 2014, the most coveted prize of all, the Fields Medal. The honour was accorded to her at the International Congress of Mathematicians 2014, held at Seoul. At the ICMs it is customary to have presentations on the works of the Fields Medal awardees. Maryam's contributions were presented by Curtis McMullen.

The towering graph of a career, with all the attendant encomiums, was lined with the dark cloud of breast cancer. She had been battling it even prior to the Fields Medal award.

Despite struggling against the frightful disease and the excruciatingly painful medical treatment she had to undergo, Maryam continued to attend conferences, workshops, seminars, lectures, and carry on with her mathematics with great enthusiasm. Two instances are worth mentioning.

One of us (RS) had the opportunity to participate in a conference in 2015, on Homogeneous Dynamics, at the Mathematical Sciences Research Institute (MSRI), Berkeley, where Maryam was also present. The lecture hall had a balcony overlooking the hall, and since Maryam was unable to sit for long due to her medical condition, she would be standing there in the balcony and following all the lectures. She would be shooting candid questions at the speakers from up there. The energy and enthusiasm with which she followed all the lectures was amazing. She also gave a talk, which was very well-received.

The other is an excerpt from an editorial written by Zahra Gooya, Member-at-Large of the Executive Committee of The International Commission on Mathematical Instruction (ICMI):

Kasra Rafi from the University of Toronto wrote that in 2016, he and Maryam walked from her house at Palo Alto to Stanford's mathematics department to attend a lecture delivered by Mikhail Gromov; in a time that Maryam's cancer spread to her bones and liver and she was in harsh pain, knowing that the time of her flight to eternity is very near. It was amazing that, even in that situation, she restlessly walked to listen to a talk! Kasra said they had to stop "every few minutes along the walk so that she could lie down on a bench to rest" (see [6]).

Maryam lost the battle against cancer on 14th July 2017. On her death, Terence Tao, a Fields medalist from 2006, wrote in his blog [7] about the advice he gave Maryam after she was awarded the Fields Medal. "As the first female recipient of the Medal, and also the first to come from Iran, Maryam was experiencing these pressures to a far greater extent than previous medalists, while also raising a small daughter and fighting off cancer. I gave her what advice I could on these matters (mostly that it was acceptable – and in fact necessary – to say 'no' to the vast majority of requests one receives)." Maryam followed Tao's webpage and cited it as a source of advice on mathematical careers (see [1]).

Maryam leaves behind her young daughter, Anahita, husband Jan, and a huge mathematical legacy, which is an inspiration for mathematicians the world over and especially women mathematicians.

2. Research

Maryam Mirzakhani has made novel contributions to several areas of mathematics, including algebraic topology, geometry, and ergodic theory. The main theme of her research is understanding the geometry and dynamics on hyperbolic surfaces. A 'hyperbolic surface' is a surface with constant negative curvature[1]. The simplest model of a hyperbolic surface in three dimensions is a hyperboloid, which is obtained by rotating a hyperbola around one of its principal axes. The genus of a torus (viz. the surface of a doughnut) is one, and a genus two surface can be visualized

[1] Unlike the plane, which is flat, or the sphere, which has positive curvature, these surfaces are negatively curved.

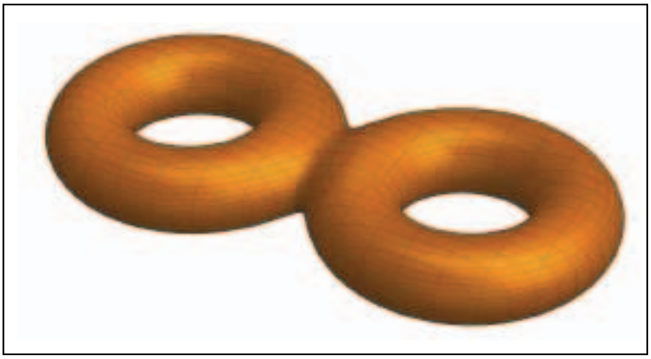

Figure 1. Genus two surface – double torus (graphics produced using Wolfram Mathematica 11).

as attaching two tori side by side, see *Figure* 1. Any surface with genus at least two is a hyperbolic surface. A 'geodesic' on a surface generalizes the notion of a straight line in a plane, and is a path on the surface joining two points that is made up of segments each of which is the shortest one joining its endpoints (e.g., a great circle on the sphere). A 'simple' geodesic is one which does not intersect itself. In contrast to the plane where the sum of interior angles of a triangle is 180 degrees, the sum of interior angles of a triangle formed by geodesics in a hyperbolic surface is less than 180 degrees. It is not difficult to check that on a sphere, which is a positively curved surface, the sum of interior angles of a triangle formed by geodesics, which are segments of great circles, is greater than 180 degrees.

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In 2004, Mirzakhani proved several path breaking results in her thesis [3]. Her thesis work was published in three papers in top mathematics journals [8, 9, 10]. Each of these works is significant in itself. One of the results she proved relates to counting the simple closed geodesics on a complete hyperbolic surface of finite area. The work of Delsarte, Huber, Selberg, and Margulis gives that the number of closed geodesics of length at most L on a hyperbolic surface grows like e^L/L, as $L \to \infty$ (known as the prime number theorem for hyperbolic surfaces). Mirzakhani [10] proved that the number of simple closed geodesics of length at most L on a hyperbolic surface X of genus g grows only polynomially, as $L \to \infty$, given by $c(X)L^{6g-6}$, where the constant $c(X)$

depends on the space X in a very 'nice' way. Thus most closed geodesics are self-intersecting.

In her thesis, Mirzakhani gave another proof of a conjecture by Witten. The first proof was given by Kontsevich [11] in 1992, for which he was awarded the Fields Medal in 1998. She also obtained several asymptotic formulas, one of which was to compute the frequencies of different types of simple closed curves. In particular, on a genus two surface, she proved that a sufficiently long simple closed geodesic is six times more likely to be non-separating than separating[2].

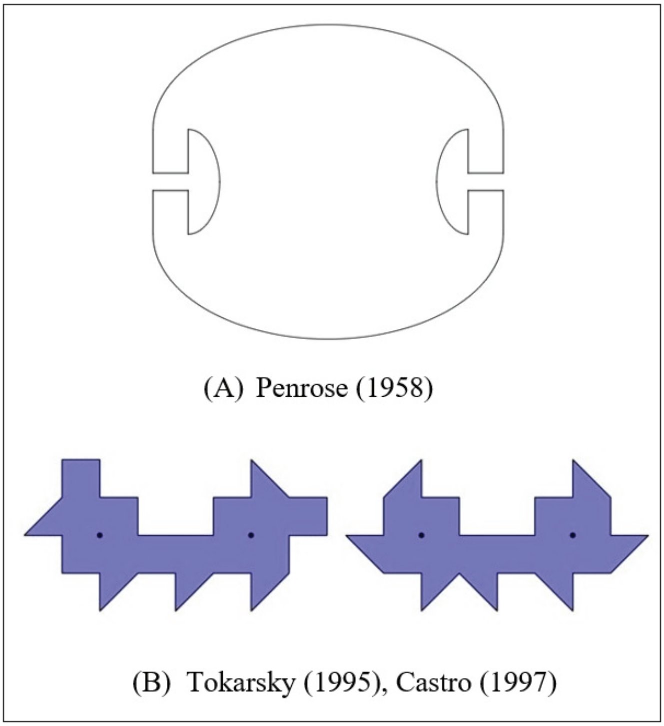
[2]A separating closed curve is one that divides the surface into two connected parts.

In a massive article with over 200 pages, Eskin and Mirzakhani [12] proved the famously known 'Magic Wand Theorem' [13] which led to the resolution of various unsolved mathematical problems. McMullen [14] had proved this theorem for genus two surfaces, but his techniques did not extend to higher genera. Eskin and Mirzakhani (and later with Amir Mohammadi [15]) proved this result for higher genus surfaces, integrating ideas from topology, geometry, and dynamical systems. They have developed new methods to study spaces of hyperbolic surfaces of any given genus. This result in higher genus has an immediate application to the famous billiards problem, which is to understand the geometry of the initial billiard using the geometry of the orbit of any trajectory on the billiard. Another application of this result is to a mathematical problem known as the 'Illumination Problem', which was posed in the 1950's by Ernst Straus. This problem can naively be stated as follows:

Imagine a room with mirrored walls. If a candle is placed at some location in the room, will it illuminate every other point in the room? (As the light ray hits the wall, it reflects back at the same angle as the angle of incidence.) In *Figure* 2(A), a curved room created by R Penrose in 1958 is shown where no matter where the candle is placed, there is a patch with positive area which is not illuminated. In *Figure* 2(B), a 26-sided polygonal room created by Tokarsky (1995), and a 24-sided polygonal room designed by Castro (1997), are shown. In both these models, exactly one marked point is not illuminated by the other marked point, even

Imagine a room with mirrored walls. If a candle is placed at some location in the room, will it illuminate every other point in the room?

Figure 2. Various models of rooms with mirrored walls.



after repeated reflections. A 'translation surface' is a finite union of polygons, with a choice of pairing of parallel sides of equal length. Such a structure arises naturally in the study of billiards and is used to frame the illumination problem mathematically. A square with opposite parallel sides identified gives a torus, and an octagon with opposite parallel sides identified gives a genus two surface, as shown in *Figure* 1. Lelièvre, Monteil, and Weiss [16] proved, drawing upon the work of Mirzakhani, that for any translation surface M, there are only finitely many points in M which are not illuminated by a given point in M. In terms of the original naively posed illumination problem, this means that if the room is polygonal with each angle a rational multiple of π, then no matter where the candle is placed, only finitely many locations in the room will not be illuminated.

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Another major contribution of Mirzakhani is in connection with the Thurston's earthquake flow. To every closed geodesic α on a hyperbolic surface X of genus g and any real number s, one can associate a new hyperbolic surface X_s of genus g using Fenchel–Nielsen construction: cut X along α, twist by length s to the right, and re-glue. Note that the resulting twist path is periodic, in the sense that $X_{s+\ell} = X_s$, where ℓ is the length of α. The earthquake flow is a natural generalization of this. Mirzakhani [17] proved a remarkable result that the earthquake flow is ergodic. This means in particular that the orbit of almost every point under the earthquake flow gets arbitrarily close to any other point. Her proof exhibited a link between the holomorphic and symplectic structures of the space of hyperbolic surfaces of a fixed genus.

Mirzakhani has left researchers with plenty of ideas and thoughts to build upon. Her novel contributions, directly and indirectly to several areas of mathematics and physics, are still unfolding. She remains truly an inspiration for researchers of this age and generations to come.

Suggested Reading

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