On the Relationship of Mathematics to the Real World

Deepak Dhar

In this article, I discuss the relationship of mathematics to the physical world, and to other spheres of human knowledge. In particular, I argue that mathematics is created by human beings, and the number π cannot be said to have existed 100,000 years ago, using the conventional meaning of the word 'exist'.

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The book (nature) is written in mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.

Galileo Galilei, in *Il Saggiatore* (1623).

The relationship between mathematics and science, where the latter is taken here to be the study of the real world, has fascinated philosophers of science for a long time. I read about some of these ideas when I was younger, of an impressionable age, and accepted them without much thought. But in the last few years, on further reading, and ruminating about this topic, I realized that what I took for granted as obvious truth then, I no longer believe in now. I want to share my new-found wisdom with the readers. These ideas are not new and have been discussed by many other people at many other times. I am reiterating them here, as they still are contrary to the prevailing conventional wisdom.

In general, the view of practicing scientists is to stay away from philosophical discussions, and the advice given to young research scholars is to 'shut up, and calculate.' In fact, philosophizing is considered bad manners and a sign of old age by many scientists. I think that some of this disrepute may be blamed on the opacity of many philosophical discussions. 'Philosophy is the misuse of a terminology, which was invented just for this purpose' [1].

Keywords
Wigner, Abbot, philosophy of science, use of mathematics in physics.

About 'shut up and calculate', many people would agree that the so-called philosophical questions are the more important ones. This article is aimed at the younger readers. I do not think one should discourage the young people from thinking about philosophical questions, just because there are no clear and unambiguous answers, or because this does not lead to a publication. Also, philosophical discussions need not be incomprehensible. I will try to keep the arguments here transparent. A discussion of philosophical issues amongst young people stimulates ideas, promote critical thinking, and may even clear misconceptions. This is my hope.

Just one more point. Often, readers of philosophical arguments have some prior beliefs, and if the writer says something they already believe in, they go "Right. Right. Right." If, on the other hand, it does not, they immediately dismiss the writer as wrong, without making any attempt to re-examine their own beliefs in the light of the arguments given. I hope that you will not do this.

1. The Everyman's View of Mathematics

The popular sentiment about mathematics is either of unadulterated hate or of awe and supreme reverence. The latter is captured in the idea that God is a mathematician, (or takes orders from a mathematician). One may find similar views expressed in well-known sayings that describe mathematics as 'the crest of a peacock' [2], or the 'queen of sciences' [3].

The first quote is from the *Vedangas*, and this suggests that this is the inherited wisdom of our sages, and was a generally accepted view at that time. However, one may ask how many people in ancient or medieval India, at any given time, could be called mathematicians, in the sense that they at least knew about Aryabhatta's or Bhaskara's work, not just their names, and could explain it to others, even if they did not write any books on mathematics themselves? Would it be of the order of 5, or 50, or 500? Most educated estimates about this number from experts tend to be nearer to 5, than to 50. Hence, it seems to me that, in spite of the quote,

Only a few people, in ancient or medieval India, living at any given point of time, could reasonably be called mathematicians.

and in spite of the well-known achievements of Indian mathematics of very high order, mathematics has not been held in such a high regard, in practice, in the Indian philosophical tradition. The second quote, from Gauss, presumably conveys truthfully what he believed, but cannot be considered unbiased. We would like to take a harder, less starry-eyed look at the relation of mathematics to other spheres of human knowledge.

Gauss considered mathematics to be the 'queen of sciences'.

To be specific, let us start with the question, "Did the number π exist $100,000$ years ago?" I suspect that a good fraction of readers is thinking "Yes. Obviously!" What I will like to argue below is that the answer is not so obvious, if you think about it a bit.

Did the number π exist $100,000$ years ago?

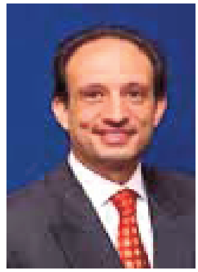
Of course, the answer depends on what we mean by 'exist'. Clearly, the number π is not a physical object, like a table, or the planet Jupiter, and it does not exist in the same sense that a table exists. A material object has mass, and occupies an identifiable region of space and time. A number like π is a concept and can exist only as such.

For example, one may speak of 'eight-headed zebras'. Such animals do not exist in the real world anywhere. But, just by putting these words together, I create these objects as a 'mental category'. It starts existing in the world of ideas. One can then deduce several properties of such objects. How many eyes does an eight-headed zebra have? The answer is sixteen, as each head has two eyes.

What is true for eight-headed zebras, is equally true for the perfect circles of Euclidean geometry. There are no perfect circles to be found anywhere in the real universe, but one can prove theorems about perfect circles from the definition, as is done in high-school geometry text-books.

In 1960, Wigner wrote a very influential article titled 'The Unreasonable Effectiveness of Mathematics in Natural Sciences' [4]. This has led to a lot of discussion amongst philosophers of science. The points made by Wigner have been elaborated upon, analyzed, and critically discussed by many others. In particular, building on the earlier discussion of Hamming, Derek Abbott, a

Eugene Wigner



Derek Abbott

professor of electrical engineering at the University of Adelaide, Australia wrote a counter-point titled 'The Reasonable Ineffectiveness of Mathematics' in 2013 [5]. I found his arguments rather persuasive, and they led me to change my position. By this article, I am trying to spread the good word.

2. The Wigner Argument

Let me start by summarizing the Wigner argument. Wigner starts his article with a story about two friends, who were classmates at high school, and then meet again after many years, and are discussing what they do. One of them has become a statistician and shows a reprint of his latest paper on population trends. The friend asks what is a normal distribution, and this is explained in terms of the distribution of heights of men in a city. He asks about the symbol π in the equation, and on being told that it is the ratio of the circumference of a circle to its diameter, he is unconvinced, and thinks the friend is putting him on, as "surely the population has nothing to do with the circumference of the circle".

Wigner argued that mathematical concepts find uses in unexpected places, and provide an unreasonably good description of physical phenomena.

Wigner comments that the reaction of the classmate only betrayed common sense. He uses this example to build his main thesis that "mathematical concepts turn up in entirely unexpected places. Moreover, they often permit an unexpectedly close and accurate description of the phenomena."

He then goes on to explain the terms 'mathematics' and 'physics'. It seems necessary to briefly discuss this here also, even at the risk of sounding pedantic, and turning off some readers. The point is that there is no agreement about what these words mean, and in addition, the meaning has evolved with time.

The meaning of the word 'mathematics' has evolved in time, and needs clarification, as it may mean different things to different people.

For example, Georg Ohm, of the Ohm's law fame, in his paper dealing with currents induced in wires by applying potential differences wrote that he believed that investigations would "secure inconvertibly to mathematics the possession of a new field of physics, from which it had hitherto been totally excluded .."[7]. This was written only about 200 years ago. But, the sentence seems rather strange to a modern reader. Ohm's law is clearly a law of physics. Since when did it become a part of mathematics? Clearly, Ohm's use of the word 'mathematics' does not conform to its present meaning. Some clarity about what we are calling 'mathematics' here is necessary. I guess that Ohm used it in the same sense as some students do, when they say that the equation: $s = \frac{1}{2}gt^2$ is mathematics, but the equivalent statement "for falling bodies, acceleration is constant" is not. This (wrongly) identifies mathematics with the use of a mathematical equation to describe the relationship between numerical measures of physical quantities.

Wigner's description of mathematics is somewhat obscure: "... I would say that mathematics is the science of skillful operations with concepts and rules, invented just for this purpose." Examples of mathematical concepts he gives are complex numbers, algebras, and linear operators. He notes that these concepts are additional ingredients to the mathematical structure. "A mathematician could formulate only a handful of interesting theorems without defining concepts beyond those contained in the axioms..".

About the 'sciences', he notes that, in general, the world around us is unpredictable, but "in spite of its baffling complexity, certain regularities can be observed". These are 'the laws of nature', for example, the Kepler's laws of planetary motion. He says that it is a miracle, and not 'natural', that 'laws of nature' exist, and are the same everywhere. Even more miraculous is the fact that man is able to discover these laws. The study of these laws is what Wigner calls 'Science'.

For Wigner, science is the study of regularities observed in the natural world.

He cites Galileo's remark given in the beginning of this article, and notes, "A physicist uses some mathematical concepts for the formulation of laws of nature, and only a small fraction of mathematical concepts is used in Physics. Mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena".

He cites as examples the calculation of ground state energy of helium, and Lamb shift calculation of quantum electrodynamics and concludes with "the miracle of appropriateness of mathematics for the formulation of laws of physics is a wonderful gift that we neither understand, nor deserve. We should be grateful for it, and hope that it will remain valid in future."

3. The Abbott Counterpoint

I now try to summarize the arguments of Abbott. Abbott starts by noting the two basically different philosophical positions that he calls Platonist, and non-Platonist.

Plato argued that or sensory perceptions are limited, and imperfect, and may not reflect the true nature of things. For example, the birds actually exist, and are three dimensional, and this fact is independent of the existence of, or the limitations of the observers.

Plato discussed the imperfectness of our sensory perceptions, and compared them to the world seen by some hypothetical cavemen, who have never been outside the cave, and can only perceive the world outside from the shadows they cast on the walls of the cave. Thus, they only see shadows of birds and not the real birds. From this example, he argued that there is a world outside, independent of our sensory perceptions, and this is the actual world, and what we perceive by our senses are only shadows. Following this line of thought, mathematical forms, like natural numbers are a part of this ideal world outside, and they have their own existence, independent of our perception.

A natural extension of this viewpoint is the idea that numbers like 7, and π were there even when mankind was not there. Note that I chose the time in the question to be 100,000 years. It is much smaller than the age of the big bang, or the age of the Earth (about 4 billion years). By that time, most of the dinosaurs were long extinct, but humans were not clearly distinguishable from other apes.

The opposing position is that our mathematics is very much a result of human cultural evolution. And, further, mathematical forms are made by people as we go along, tailoring them to describe reality. For a non-Platonist, there is no perfect circle, anywhere in the universe, and π is merely a useful mental construct.

Mathematics is a result of human cultural evolution, and the mathematical descriptions we use, may be unreasonably effective, but are still only approximations.

Abbott says that in his experience, about 80% of mathematicians are Platonist, while engineers typically tend to be non-Platonists. Physicist, he says, are often 'closet-non-Platonists': in public, they side with the Platonist position, but are unsure of it, in their hearts.

Regarding the unreasonable effectiveness, Abbott's view is that mathematics is not very successful in most real-world problems, and the apparent effectiveness is a result of focussing only on the cases where it works. Mathematics is much less successful in describing biological systems, and even less in describing economic and social systems. We have cherry-picked a few successful cases, out of a large number of much less fortunate ones. Mathematics can appear to have an illusion of success if we are preselecting the subset of problems for which we have found a way to apply mathematics.

About the Kepler problem, discussed by Wigner, he says that it is a self-selected example, and relies on our fascination with squared numbers. Actual orbits are elliptical only to finite accuracy, and, in any case, the Newtonian theory is only an approximation to general relativity. While the elliptical orbit is a very good approximate description, this is not an example of Mathematics miraculously letting us arrive at the true nature of things.

The non-Platonist position is that mathematics is a product of human imagination. All our physical laws are based on some simplifications/ idealizations/approximations, and hence are imperfect. Mathematics is a human invention for describing patterns and regularities. It follows that mathematics is a useful tool for describing the regularities we see in the universe.

4. My Own Position

If mathematical truths are independent of the existence of humans, we could ask for role of mathematics in the world of other animals.

If mathematics is independent of human existence, we can try to see what role mathematics played in the world, when humans were not there, as seen by other animals, say worms, frogs, birds, or even dogs, and monkeys. This is the reason for the choice of 100,000 years as the time in the question. In the case of bacteria or worms, it is hard to see what role mathematics has in their world. Birds have bigger brains, and form bonds with mates. Some bird-species are known to show disturbed behavior if some eggs are removed from their nest, which is evidence that they can distinguish between two or three eggs in the nest, and thus can count up to three or four. It is hard to imagine mathematics playing any more significant role than this in the mental or worldly life of birds.

In discussing the role of mathematics in the world of other animals, it seems useful to distinguish between different levels of mathematics.

The zeroth level, that I will call pre-mathematics, is the innate sense of numbers and shapes, that we share with other animals. This is the result of evolution. It helps animals move about in their environment, catch prey, or evade predators. It is clearly very effective in this. I think it is fair to say that mathematics used by other mammals like dogs, and horses, even monkeys does not go substantially beyond this level.

The first level will be the math that is expected to be known to students passing out of primary school. It consists of some familiarity with simple operations with integers, or fractional numbers, how to add them, or multiply, etc., not much more.

The second level, which I will call high-school level mathematics, involves the use of symbols, the notion of proof, and abstractions like $\sqrt{2}$. This is the kind of math we teach children in high school, and this is all the math that even humans knew, even as late as a few thousand years ago. This is necessary for commerce and engineering. In buying and selling, we (mostly) need to know only the addition and multiplication of numbers. To make buildings that do not fall down, we need to use concepts from geometry, and strengths of materials, used in setting up buildings, can be expressed in simple power-law relations between load and size, etc. This uses level-2 math. This is also very effective, and perhaps it is not surprising, or 'unreasonable' that this is so.

The third level, which I will call higher mathematics, is what is not covered in the first two levels. Clearly, there are no sharp boundaries between these levels. For example, I chose not to include calculus in level-2 but could have.

A biologist will note that there is no credible evidence of behaviour in non-human animals showing that they have the mental capability of dealing with any level-1 mathematics. So, I am not sure of what mathematics we could be talking about in a world without humans [8]. If that sounds too anthropo-centric, let me add that this is not a question of humans versus non-humans. Even amongst humans, mathematical ability is not very uniformly distributed. At a conservative estimate, about half of the population of high-school students, in all countries, have serious difficulty with level-2 math. I should add that this does not seem to seriously affect their ability to enjoy life or contribute to the society.

Being able to detect patterns, approximate, and idealize is a capability that we, as a species, have learned through evolution. This capability is rather basic, and an example is the well-known visual illusion known as the Kanisza triangle (see *Figure* 1). It is clearly useful to be able to make sense out of the noisy and incomplete sensory data that we get. This ability presumably also exists in other animals. Clearly, it helps in survival if you can detect a predator, partly hidden in the grass. Seeing ellipses in planetary orbits is an example of the same idealization/ approximation/filtering process. However, it would not be correct to equate this ability with 'mathematics'. It seems to me that 'unreasonable effectiveness' that Wigner is talking about is mainly about mathematics of the third level. In areas like biology, psychology, and even geology, mathematics is not particularly useful. So, let

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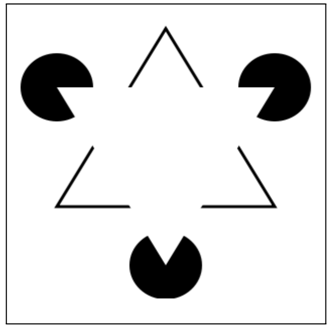

Figure 1. The Kanisza triangle. We tend to see a well-formed inverted triangle, which is actually not there. Taken from https://explorepsychology.wordpress.com/2011/11/25/illusions/

us agree that he is mainly thinking about physics, even when he speaks of the 'sciences'. Even in physics, in areas where one would expect math to be effective, like predicting the motion of a cricket ball, you do not have to be a Tendulkar, or Kohli, to realize that, in the real play, it is not. I find fully convincing Abbott's argument that the 'effectiveness' of mathematics is a result of making the scope of regime of application very limited, only to questions where it is effective. Then mathematics is effective, but expectedly!

To come back to the equation $s = \frac{1}{2}gt^2$ discussed earlier, one important implicit assumption in this equation is that distance moved may be represented as a real number. One may ask if the 15th place in the decimal representation of distance moved expressed in cms, exists, in any real sense, for a macroscopic body falling in air. Note that we are not talking of Planck length scales (10^{-35} meters, where quantum gravity effects studied in string theory come into play), but this is still about 10^{-5} of the size of an atom. A bit of thought will show that actually, even the concept of center-of-mass is not well-defined at this level of precision. All the atoms in the ball jiggle, due to thermal motion, with amplitudes of order 10^{-8} cm. Also, some molecules would be getting rubbed off, as the ball moves through the air, and I am not sure if their position should be included in the calculation of the center-of-mass. This again underlines the point that we simplify, idealize the actual problem, and describe the height of the falling ball by a single real number, and only then the simplified problem becomes tractable using some mathematical tools.

That mathematics deals only with a small set of possible scientific questions was well-appreciated by Wigner. He wrote, in the same article:

All the laws of nature are conditional statements that permit prediction of some future events ...(like position of the planets) on the basis of some aspects of the present state... As regards the present state of the world, such as the existence of the planet earth on which we live, and on which Galileo's experiments were performed, the existence of the sun, and all our surroundings, the laws of nature are entirely silent.

On the meaning of 'existence' in the Platonic world: our conviction that a given table 'exists' arises out our experience of seeing it, and feeling it by touch, etc. These direct sensory experiences may nowadays be augmented by more sensitive instruments like microscopes, X-ray cameras, chemical sensors, or particle-colliders, if needed. It is the world of real objects like tables and rivers, and Higgs bosons, that science deals with. Plato has used a very misleading analogy and assigns cavemen's guess of what the shadows might be a higher level of existence than to the shadows themselves. We may even be willing to assign a higher status to the idea of a table, than to the table itself. But does that make the former more real?

We may be willing to assign a higher status to the idea of a table, than to the table itself. But does that make the former more real?

5. Concluding Remarks

A discussion of the relationship between mathematics and physics, is not complete without some mention of the interactions between mathematicians and physicists (in their professional capacities). Here, I will like to retell my favorite story, that I read first in an article by C N Yang (unfortunately I have been unable to track the original article). It concerns a physicist who is travelling across the USA, attending conferences, and giving lectures at different places. He arrives in a small university town, checks into a hotel and is walking up and down unfamiliar streets, with a bag of laundry, looking for a place where he can wash them. After a long walk, finally, he finds a shop with a signboard 'Laundry done here'. He is much relieved, enters the shop, and puts his big bag of laundry on the counter. On the other side, is an oriental-looking man. The man seems mildly annoyed, looks up at the physicist, and asks,"What you want?". The physicist is a bit angry. He says,"I want my laundry done." "No laundry here." The physicist objects, and points to the signboard, "But see, it says right here, that laundry is done here." The man behind the counter smiles,"Ah, that! We make signs."

I like the story because it captures the frustration of physicists in trying to get a mathematician to help them in their work. Quite often, a mathematician would find the problem the physicist wants to address uninteresting. Of course, the reverse is also true. For example, a mathematician would worry about the existence and uniqueness of a solution, which a physicist is quite happy to take for granted. Certainly, in most cases, the driving force behind a mathematician's work is not its usefulness to science.

To return to the question we posed at the beginning of the article: "Did π exist 100,000 years ago?" I have tried to convince you that the answer has to be 'No'. A concept may be said to exist, after the first time someone thought of it, but even before that? What is true about π, also holds for other more sophisticated mathematical constructs. All the mathematics we know is made by humans, and the same holds for mathematical concepts. For a more detailed discussion of this broader thesis from different perspectives, I can recommend the collection of essays [9], available on the internet. I will end by quoting two sentences from the essay by S Wenmackers in the collection [10]:

A concept is not like an object like Jupiter, which we can believe existed, even before it was first observed. So mathematical constructs like π can be said to have come into existence only after they have been thought of the first time.

The fact that our so-called laws can be expressed with the help of mathematics should be telling, since it is our (emphasis in original) science of patterns. When we open Galileo's proverbial book of nature, we find it is full of our own handwriting.

Suggested Reading

[1] **W Dubislav, *Die Philosophie der Mathematik in der Gegenwart* (Berlin: Junker and Dunnhaupt Verlag, 1932), p.1, as quoted in [4].**

[2] "Like the crest of a peacock, like the gem, on the head of a snake, so is mathematics at the head of all knowledge.", translated from *Vedanga Jyotisa* (c.500 BC).

[3] C F Gauss, as quoted in Gauss zum Gedächtniss (1856) by Wolfgang Sartorius von Waltershausen: Mathematics is the queen of sciences and number theory is the queen of mathematics. source: *Wikipedia*.

[4] E Wigner, Unreasonable Effectiveness, https://plus.maths.org/content/unreasonable-effectiveness

[5] D Abbott, Reasonable Ineffectiveness, https://pdfs.semanticscholar.org/462d/7b6b1ee8243b6aa8897be3cf306239fb43c6.pdf

[6] Photos taken from http://www.nobelprize.org/ nobel_prizes/physics/laureates/1963/wigner_postcard.jpg and http://telsoc.org/event/sa/2014-03-18/ forensics_engineering

[7] G Ohm, *The Galvanic Circuit, Investigated Mathematically*, Ed. Richard Taylor (New York, Johnson Reprint of the 1841 edition, 1966) p.404, Originally published in German, in 1827.

[8] Discussion about math that is possibly used by the klingons from another planet will be left for science-fiction fans.

[9] *Trick or Truth: The Mysterious Connection Between Physics and Mathematics*, Eds. Anthony Aguirre, Brendan Foster and Zeeya Merali, Springer, 2016.

[10] S Wenmackers, *Children of the Cosmos*, in [9].

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