This article studies how the height of water varies with time when water flows out through a horizontal tube at the bottom of a cylindrical bottle. As an extension of the study, this article also studies the flow through two bottles. This experiment is exciting as well as thought-provoking.

1. Introduction

This is a simple exercise in elementary fluid dynamics for the undergraduate and the secondary school level. Here, we explore the flow of water through an orifice at the bottom of a cylindrical bottle/tank, first through a tube attached to the bottom of the bottle/tank and then without the tube. The experiment is easy to perform. The highlight of this experiment is the use of Tracker software which is freely available and is a great video analysis tool. The article is organised as follows:

- Using Bernoulli’s equation to arrive at an expression for the velocity of efflux of the fluid from the bottom of the cylindrical tank, followed by the introduction of viscosity into the equation.

- Derivation of an expression for the variation of height of the liquid column with time in one-bottle and two-bottle systems.

- Experiment using a one-bottle system with a small bore tube attached to the bottom of the bottle.

- Experiment using a one-bottle system without the tube.

- Experimental analysis of flow of liquid from one bottle to another and efflux from the second bottle.

Keywords
Bernoulli’s equation, fluid dynamics, efflux, viscosity, turbulence.

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Bernoulli’s equation\(^2\) is essentially the conservation of energy equation for fluids; corresponding to this is Bernoulli’s effect which states that the pressure gets lowered when the speed of the fluid is increased. Daniel Bernoulli was a Swiss scientist who published this in the year 1738. It is notable that the flight of aircrafts is based on this equation. Here, Bernoulli’s equation is used to derive the relationship between the height of a liquid column and the distance to which the outflow is projected when allowed to flow through a small orifice. Then, this is experimentally verified using traditional lab equipment, incorporating use of information and communication technology (ICT) throughout the study. This was done in order to improve the accuracy, precision, and simplicity. The apparatus used were simple and inexpensive so that the experiment is easily replicable.

2. Bernoulli’s Equation and Velocity of Efflux

This section deals with the derivation of the relationship between the distance to which a jet of water is projected when it flows out of an orifice at the bottom of a cylindrical bottle/tank using Bernoulli’s equation.

\[
P + \rho g y + \frac{1}{2} \rho V^2 = \text{constant}. \tag{1}\]

Applying this to the system at points ‘A’ and ‘B’ in Figure 1:

\[
\frac{1}{2} \rho V_a^2 + \rho g y_a + P_o = \frac{1}{2} \rho V_b^2 + \rho g y_b + P_o, \tag{2}\]

where \(\rho\) is the density of the fluid, \(V_a\) and \(V_b\) are the speed of the fluid at ‘A’, and ‘B’ using \(H(t) = (y_a - y_b)\), where \(y_a\) is the height of the level of water from the bottom of the cylinder, and \(y_b\) is the the distance of the horizontal tube from the bottom of the cylindrical tank. \(H(t)\) is the level of the fluid from the horizontal tube as a function of time and \(P_o\) is the atmospheric pressure. Cancelling out \(P_o\) and \(\rho\) we get:
\[ \frac{1}{2}(V_b^2 - V_a^2) = gH(t). \]

To eliminate \( V_a \), we use the equation of continuity, which is \( V_a A = V_b a \), where \( A \) is the cross-sectional area of the cylindrical tank and \( a \) that of the tube at the bottom of the tank.

\[ V_a = \frac{V_b a}{A}. \]

Using this we get,

\[ V_b^2 = \frac{2gH(t)}{1 - \frac{a^2}{A^2}}. \]  
(3)

\[ V_b^2 = \frac{2gH(t)}{1 - \frac{r_2^2}{r_1^2}}. \]  
(4)

Since in this case, a cylindrical bottle of radius \( r_1 = 3.9 \) cm and an outlet pipe of radius \( r_2 = 0.225 \) cm were used:

\[ \frac{r_2^2}{r_1^2} \approx 0.057 \approx 0. \]

Hence,

\[ V_b^2 = 2gH(t). \]  
(5)

The behavior of the water efflux can be modeled around a simple projectile motion system. For projectile motion, acceleration...
along x-axis is 0. As the initial velocity $u = 0$ along the y-axis, its component distance is given by:

$$y = \frac{1}{2} gt^2. \quad (6)$$

Hence distance is given by,

$$x = V_b t,$$

$$t = \frac{x}{V_b} = \frac{x}{\sqrt{2gH(t)}}. \quad (7)$$

Substituting (7) in (6) we get an expression for $Y_o$ which is the vertical distance from the tube to the scale which is the point of measurement of x.

$$Y_o = \frac{1}{2} g \left( \frac{x}{\sqrt{2gH(t)}} \right)^2, \quad (8)$$

which yields,

$$X^2 = 4Y_oH(t). \quad (9)$$

Therefore when plotting $X^2$ versus $H(t)$, the gradient is $4Y_o$.

3. Modification of (9) Using the Hagen–Poiseuille Equation

Equation (9) gives the range to which a liquid gets projected from the bottom of a bottle through a tube of length l, neglecting the viscosity effects. But the diameter of the tube is small and viscosity effects cannot be neglected. Therefore (9) needs to be modified using the Hagen–Poiseuille equation. Viscosity is the internal friction of a fluid – within itself and between the walls of the tube, restricting free flow. The Hagen–Poiseuille equation is given by:

$$\phi(t) = \frac{dV(t)}{dt} = \frac{AdH(t)}{dt} = \frac{\pi \Delta PrA}{8\eta l}. \quad (10)$$

Here, $\phi$ gives the rate of flow of water and $V(t) = AH(t)$ gives the volume of water which changes with time. $\Delta P$ is the pressure difference which is $\rho g H(t)$, and $\eta$ is the coefficient of viscosity. It should be noted that the Hagen–Poiseuille equation can only be applied for laminar flow and not for low viscosity fluids.
be applied for laminar flow. Thus it cannot be applied for low viscosity fluids because the liquid would start exhibiting turbulent flow.

Using $\Delta P = \rho g H(t)$ in (10) and rearranging we get:

$$\frac{AdH(t)}{H(t)} = \frac{\rho g \pi r^4}{8\eta l} dt.$$ (11)

Integrating the above equation between the limits $H_o$ (the initial height (at $t = 0$) of the water in the tube), and $H$ (the height of the water at time $t$), we get:

$$\int_{H_o}^{H} \frac{AdH(t)}{H(t)} = \int_{0}^{t} \frac{\rho g \pi r^4}{8\eta l} dt.$$ (12)

Let,

$$k = \frac{\rho g \pi r^4}{8\eta l}.$$ (13)

Hence (12) becomes,

$$A \int_{H_o}^{H} \frac{dH(t)}{H(t)} = k \int_{0}^{t} dt.$$ (12)

On integration we can get the expression for $H(t)$

$$H(t) = H_o e^{-\frac{k}{A} t}.$$ (14)

The above expression shows the exponential decay of the height of the liquid with time. Many natural phenomena depend on exponential functions such as population growth, radioactivity etc. Substituting (9) in (14) we get:

$$X^2 = 4Y_o H_o e^{-\frac{k}{A} t}.$$ (15)

Taking log on both sides of (14) we get:

$$\ln(X) = \ln(4Y_o H_o) - \frac{k}{2A} t.$$ (16)

(16) is of the form $y = mx + c$. On plotting $t$ against $\ln(X)$ it should give a straight line with slope $-\frac{k}{2A}$ and y-intercept $\frac{\ln(4Y_o H_o)}{2}$. (16)
or (15) gives the range of the liquid projected from a bottle through a tube of length \( l \) taking into effect the viscosity also. The gradient and y-intercept of the plot between \( \ln(x) \) and \( t \) is given by:

\[
\text{gradient} = \frac{\pi r^4 \rho g}{16 \pi \eta l}
\]

\[
y - \text{intercept} = \frac{4Y_o H_o}{2}.
\]

4. Relationship Between Height of the Liquid Column and Time in One-bottle and Two-bottle Systems Without Horizontal Tube at the Bottom

Consider two bottles arranged as in Figure 2. Water from the top tank empties into the lower tank. Thus the level of water keeps decreasing in the top tank and increasing in the lower tank. Let \( h(t) \) be the height of water in the top cylinder and \( y(t) \) be the level of water in the lower tank.

In this section, we theoretically analyse how the height of the liquid in the bottle \( h(t) \) varies with time \( t \). For that we use the equation of continuity. We are assuming that the flow is streamlined and non-viscous. This assumption is safe because the liquid is not ejected through a tube but only through an orifice at the
bottom of the cylindrical bottle.

\[ av = AV \]  

(19)

Using

\[ v = \sqrt{2gh(t)}, \]

from (5) and

\[ V = \frac{dh(t)}{dt}, \]

we get,

\[ a \sqrt{2gh(t)} = A \frac{dh(t)}{dt}. \]  

(20)

Substituting

\[ k = \frac{a \sqrt{2g}}{A}, \]  

(21)

and rearranging (20)

\[ kdt = -\sqrt{h(t)}dh(t), \]

\[ \int kdt = -\int (h(t))^{-\frac{1}{2}}dh(t), \]

which gives

\[ kt = -2\sqrt{h(t)} + c. \]  

(22)

Using initial value conditions, we can get the value of \( c \). At time \( t = 0, h(t) = h_o \) (\( h_o \) is the initial level of the water). Hence,

\[ c = 2\sqrt{h_o}. \]

Substituting in (22) we get,

\[ \sqrt{h(t)} = \sqrt{h_o} - \frac{kt}{2}. \]  

(23)

Equation (23) can be used to determine how the height of water in the second bottle \( y(t) \) varies with time when water from the first bottle flows into the second. The volume of water flowing per second from the first bottle to the second bottle is \( a \sqrt{2gh(t)} \), and the outflow per second through the hole at the bottom of the
second bottle is \( a \sqrt{2gy(t)} \). \( y(t) \) is the height of water in the second bottle. The water left in the second bottle after a time \( t \) is given by (the diameter of both the orifices in both the bottles are the same (i.e.,) \( a \)):

\[
(a \sqrt{2gh(t)} - a \sqrt{2gy(t)})t = Ay(t). \tag{24}
\]

Using (21) and (23) in (24) we get,

\[
kt \sqrt{h_o} - \frac{k^2t^2}{2} = y(t) + kt \sqrt{y(t)}. \tag{25}
\]

Using the method of completing the squares, \( y(t) \) can be written as:

\[
y(t) = \left( \sqrt{kt \sqrt{h_o} - \frac{k^2t^2}{4} - \frac{kt}{2}} \right)^2. \tag{26}
\]

The following are the theoretical results obtained:

- The variation of height of the liquid column with time in a cylindrical container when the water effluxes through a narrow tube at the bottom is given by:

\[
H(t) = H_o e^{-\frac{kt}{4}},
\]

(refer (14)).

- When we add two cylinders, the equation gets more complex and the height of the liquid in the lower cylinder varies polynomially as shown below (the plot of the graph is also shown in Figure 7):

\[
y(t) = \left( \sqrt{kt \sqrt{h_o} - \frac{k^2t^2}{4} - \frac{kt}{2}} \right)^2,
\]

(refer (26)).

5. Experimental Procedure and Results

5.1 Making the Bottle

- Take a standard PVC drain pipe of diameter 3 inches and cut it to about 40 cm (as shown in Figure 3).
• Close one of the open ends using a cover and stick the cap tightly in place.

• Using a power drill, make a hole near the closed end.

• A rectangular white piece of paper is stuck in onto the surface of the PVC pipe.

• Insert a body of a pen of similar diameter as the orifice; fix it using M-Seal.

• Make another hole along the same level, this time slightly larger.

• Insert a transparent tube of similar diameter and stick in place using M-Seal (or any adhesive).

• Let the pipe still for about 30–45 minutes for the M-Seal to dry.

• Repeat the above steps and make another bottle.

5.2 The One-Bottle Experiment

• Set up the PVC bottle on a stand and ensure that it is stable.
Table 1. Values used in the one-bottle system experiment.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0$</td>
<td>$(25.2 \pm 0.1) \times 10^{-2}$ m</td>
</tr>
<tr>
<td>$Y_0$</td>
<td>$(21.1 \pm 0.1) \times 10^{-2}$ m</td>
</tr>
<tr>
<td>$L$</td>
<td>$(14.2 \pm 0.1) \times 10^{-2}$ m</td>
</tr>
<tr>
<td>$R$</td>
<td>$(0.225 \pm 0.001) \times 10^{-2}$ m</td>
</tr>
<tr>
<td>$g$</td>
<td>$9.8$ m/s</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$0.911 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$1000$ Kg/m$^3$</td>
</tr>
</tbody>
</table>

Figure 4. Exponential decay of height of water in the cylindrical container.

- Take about 2 litres of water in a bucket, add a few crystals of potassium permanganate, and stir the water to get a deep purple solution.

- Close the orifice of the bottle tightly and fill the bottle to the brim with the coloured water. A bucket must be kept below the orifice in order to collect the water that flows out of the bottle.

- Keep a meter scale on a stand near the bottle.

- Set up a camera (a cell phone camera will do) on a stable mount/stand, adjust the focus and zoom to get the entire visual (including the scale) in one frame. Start the recording and then open the orifice.

For this experiment, the values indicated in Table 1 were used. The exponential decay of the height of water in the cylinder is shown in Figure 4. To linearise this graph, the log graph is also shown in Figure 5.

$H$ vs., $X^2$ graph for the one-bottle experiment is shown in Figure 6 without the tube attached to the orifice the gradient=0.0555 and y-intercept=3.914 which is in good agreement with the theoretical
values of gradient=0.0798 and y-intercept=3.829. The theoretical values were arrived at after substituting the values of Table 1 in (17) and (18).

5.3 The Two-bottle Experiment and Evaluation

After completing the experiments with one-bottle systems, we decided to explore two-bottle systems. The experiments were exactly like the first one, with the only difference being that a filled bottle drained into an empty bottle, before its efflux was recorded.

The values used in the experiment are indicated in Table 2.

<table>
<thead>
<tr>
<th>$H_0$</th>
<th>$r$</th>
<th>$g$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(25.2 \pm 0.1) \times 10^{-2}$ m</td>
<td>$(21.1 \pm 0.1) \times 10^{-2}$ m</td>
<td>$9.8$ m/s</td>
<td>$(3.9 \pm 0.1) \times 10^{-2}$ m</td>
</tr>
</tbody>
</table>

Table 2. Values used in the two-bottle system experiment.
Figure 7. Theoretical (26) and experimental variation of height of liquid in the second bottle with time.

A plot of the variation of the height of the liquid in the second bottle with time is plotted in the curve in Figure (7). The theoretical variation is plotted using (26).

Performing the two-bottle test was not as easy as one-bottle experiment due to the complexities involved such as ensuring that the water from the first bottle completely drains into the second and adjusting the positions of the bottle with changing efflux. In spite of that, the overall experiment was conducted largely free of systematic errors, which is reflected in the fact that the theoretical and experimental curves have very similar shapes. Yet, there were certain limitations to the experiment.

The theory behind this experiment assumes perfect conservation of energy. Yet in reality, a lot of energy of the efflux is lost when the outflow hits the funnel. This loss of kinetic energy of outflow is also the reason why the actual height (proportional to the velocity of the efflux) is less than the theoretical prediction. Most of this energy was lost in the form of sound and turbulence.

The theory assumes the flow to be perfectly streamlined without any turbulence. However, the changes in velocities (that were accounted to energy losses) also resulted from the turbulence that was visible in the system. Turbulence is calculated using the
Reynolds number,

\[ R = \frac{\rho vr}{\eta} = 2444.991. \]  

(27)

This means that the flow is borderline turbulent. Hence, energy loss due to turbulence is minimal but not insignificant.

6. Conclusion

This experimental study was conducted as part of the curriculum of the International Baccalaurate Diploma Programme (IBDP). The study aims at helping students understand the physics of fluid dynamics and also help them analyse and interpret the graphs obtained. The use of ICT is also brought out. The experiment is easy to perform and does not require expensive equipment.

Suggested Reading
