

# Pythagorean Means and Carnot Machines

When Music Meets Heat

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This article explores some interesting relations between Pythagorean means (arithmetic, geometric, and harmonic means) and the coefficients of performance of reversible Carnot machines (heat engine, refrigerator, and heat pump).

## 1. Introduction

Mean is an important concept in mathematics and statistics. We will consider the Pythagorean means, defined as follows:

$$A(x_1, x_2, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i, \quad (1)$$

$$G(x_1, x_2, \dots, x_n) = \sqrt[n]{\prod_{i=1}^n x_i}, \quad (2)$$

$$H(x_1, x_2, \dots, x_n) = n \left( \sum_{i=1}^n x_i^{-1} \right)^{-1} \quad (3)$$

which denote the arithmetic mean, the geometric mean, and the harmonic mean<sup>1</sup>, respectively. Here,  $\{x_i | i = 1, 2, \dots, n\}$  represents a set of positive, real numbers. Historically, these means are attributed to the ancient Greek mathematician Pythagoras, who found their use in representing ratios on a musical scale (see Box 1).

There are some common properties satisfied by the means  $M(x_1, \dots, x_n)$ :

(i)  $\min(x_1, \dots, x_n) \leq M(x_1, \dots, x_n) \leq \max(x_1, \dots, x_n)$ .  
So,  $M(x_1, \dots, x_n) = M(x, \dots, x) = x$ .

(ii) A mean is homogeneous function of degree one:  $M(\lambda x_1, \dots, \lambda x_n) = \lambda M(x_1, \dots, x_n)$ , for all permissible real  $\lambda$ .



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### Keywords

Arithmetic mean, geometric mean, harmonic mean, Carnot machine, Pythagorean means, musical scales, coefficients of performance.



**Box 1. Music in the Means**

In ancient Greece, like during the days of Pythagoras (circa 6th century B.C), the mean (of two numbers) represented a kind of balance between the two opposites of high and low. The three means  $A$ ,  $G$ , and  $H$  were regarded as forming a triad and were considered to be fundamental. The followers of the Pythagorean school were enthused all the more to find these means reflected in the order and harmony of nature. According to legend, one day, as Pythagoras was walking past a blacksmith's shop, he heard a harmonious ringing of hammers. On closer observation, he found that when the hammers with masses in a specific proportion, were sounded, the sound appeared musical to human ears. Fascinatingly, he found these proportions to be given in terms of whole numbers, such as  $12 : 9 = 8 : 6$ . Here 9 is the arithmetic mean of the extreme numbers 12 and 6. Similarly, 8 is the harmonic mean of 12 and 6. In general, for two numbers  $a > b$ , it implies the ratio:

$$a : \frac{a+b}{2} = \frac{2ab}{a+b} : b.$$

These proportions form the basis for tuning theory in music and represent the ratios of frequencies. For example, here 6 and 12 represent the fundamental and the octave. Then the arithmetic mean frequency gives the perfect fifth, and the harmonic mean frequency gives the perfect fourth. Other intervals on musical scale can also be characterized in a similar way.

The inequalities between various means are useful in the proofs of various physical laws.

(iii) A mean is called 'symmetric', if it is invariant under permutations of the indices. Thus,  $M(x_1, \dots, x_n) = M(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ .

Now, for a given set of numbers, when at least two numbers are distinct, there exist elementary inequalities between the means, for example:  $H \leq G \leq A$ . To give commonplace examples of the occurrence of Pythagorean means, suppose that you complete two legs of a journey, spending equal time  $t$  in each of them, and traveling with average speeds of  $v_1$  and  $v_2$ , then your average speed for the total trip is equal to  $H(v_1, v_2)$ . On the other hand, if you cover equal distances in the two legs (e.g. for a return journey), at average speeds of  $v_1$  and  $v_2$ , then the average speed for the total trip would be  $A(v_1, v_2)$ . On the other hand, the geometric mean may be seen as the compound average growth rate,  $G(r_1, r_2) = \sqrt{r_1 r_2}$ , over a principal amount which gets the growth rates of  $r_1$  and  $r_2$ , in the first and the second years, respectively. The inequalities between various means are useful in the proofs of various physical laws. For their applications in thermodynamics, see *Box 2*.

<sup>1</sup>See Prithwijit De, The Arithmetic Mean–Geometric Mean–Harmonic Mean: Inequalities and a Spectrum of Applications, *Resonance*, Vol.21, No.12, pp.1119–1133, 2016.



**Box 2. Means and Thermodynamics**

There are various proofs, including geometric ones, of simple inequalities between the means, such as the arithmetic mean–geometric mean inequality:  $A \geq G$ . Thermodynamic models are especially useful in giving a kind of ‘physical’ proofs for these inequalities. Thus, consider  $n$  systems with a constant heat capacity  $C$  and initial temperatures,  $\{T_i | i = 1, \dots, n\}$ . Upon mutual thermal contact, these systems arrive at an equilibrium with a common final temperature, say  $T_f$ . From the first law of thermodynamics, the initial and final energies are equal, so that:

$$\sum_{i=1}^n C(T_i - T_f) = 0,$$

which implies  $T_f = \sum_i T_i/n$ . The total entropy change during the process is:

$$\begin{aligned} \Delta S &= \sum_{i=1}^n \int_{T_i}^{T_f} \frac{C}{T} dT \\ &= nC \left[ \ln T_f - \ln(\prod_i T_i)^{1/n} \right]. \end{aligned}$$

So, by virtue of the  $A$ – $G$  inequality, we get  $\Delta S > 0$ . Arnold Sommerfeld posed this problem in his *Lectures on Theoretical Physics*, Vol. 5, as an application of the second law to prove an algebraic inequality, *i.e.*, if we assume the validity of the macroscopic form of second law ( $\Delta S > 0$ ), then the  $A$ – $G$  inequality follows. P T Landsberg also endorsed the viewpoint that herein the truth of pure mathematics is directly accessible from the principles of science. It is understandable if some authors regard mathematical inequalities to be more fundamental, something that exists prior to scientific concepts. At best, the thermodynamic or physical models may provide a kind of heuristic which makes these inequalities seem plausible.

In this article, we shall observe a natural instance of Pythagorean means in the analysis of coefficients of performance (COP) of ideal thermodynamic machines, in particular, heat engines, refrigerators, and heat pumps. It is assumed that we are provided with heat reservoirs at various temperatures representing the set of real, positive numbers over which these means may be defined, and we can run these machines by coupling them to the reservoirs.

**2. Performance Measures for Carnot Machines**

We shall first discuss the case of Pythagorean means for two numbers in the context of heat engines, refrigerators, and heat pumps. We start by defining the COPs for these machines.



A refrigerator with an irreversible operation has a lower COP, implying that it requires a larger amount of work to extract the same amount of heat from the cold system, as compared to the reversible case.

A 'heat engine' is a device that converts heat into work, as the heat flows from a reservoir at a constant hot temperature  $T_h$  to one at a cold temperature  $T_c$ . The COP for an engine is called its efficiency ( $E$ ) which is defined as the ratio of the work generated ( $W$ ) to the heat absorbed ( $Q_h$ ) from the hot reservoir,  $E = W/Q_h$ . For an ideal or reversible heat engine which does not increase the entropy of the universe, the efficiency is maximal called the 'Carnot efficiency', which depends only on the ratio of the two temperatures:

$$E_{hc} = \frac{T_h - T_c}{T_h}. \quad (4)$$

A 'refrigerator' is a device that helps to cool a system at temperature  $T_c$ , by extracting heat from it ( $Q_c$ ) and dumping this heat into a warm environment at  $T_h$ . Due to the constraint of the second law of thermodynamics, this process cannot proceed spontaneously, and so external work ( $W$ ) has to be spent to transfer the heat. The COP of refrigerator is defined as  $R = Q_c/W$ , which for an ideal refrigerator is given by:

$$R_{hc} = \frac{T_c}{T_h - T_c}. \quad (5)$$

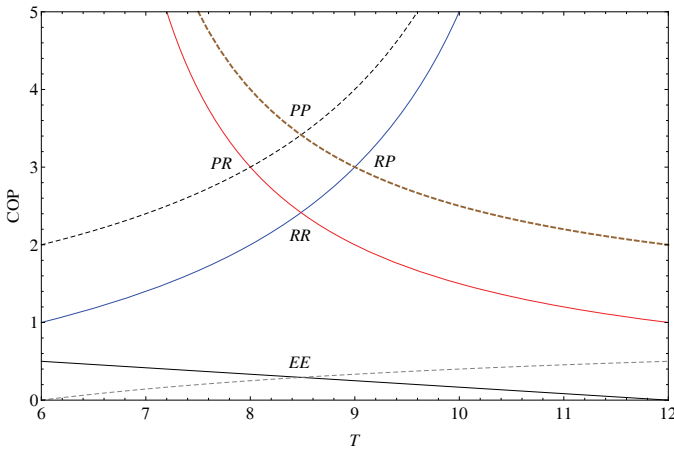
A refrigerator with an irreversible operation has a lower COP, implying that it requires a larger amount of work to extract the same amount of heat from the cold system, as compared to the reversible case.

A 'heat pump' also transfers heat from a cold to hot system at the expense of external work, but in this case we are interested in warming the system (such as interiors of a building in winter) by extracting heat from the cold environment. So, the COP of a pump is defined as the heat dumped into the warm system ( $Q_h$ ) per unit work expended ( $W$ ) for the purpose. Again, for an ideal pump, the COP is given by:

$$P_{hc} = \frac{T_h}{T_h - T_c}, \quad (6)$$

where  $T_h$  is the temperature of the interiors, and  $T_c$  is the ambient temperature. Note that, by definition, for finite temperatures,  $0 < E < 1$ ,  $R > 0$ , and  $P > 1$ .





**Figure 1.** COPs for Carnot engines, refrigerators, and pumps, between one of the extreme temperatures  $T_h = 12$ , or,  $T_c = 6$ , and an intermediate temperature  $T$ . The COPs for *EE* (as well as *RR* and *PP*) case intersect at  $T = G(T_h, T_c) = 8.485$ ; the intersection point for the *PR* case is at  $T = H(T_h, T_c) = 8$ , while that for *RP* case, is at  $T = A(T_h, T_c) = 9$ .

Configuration	Mean $T_i$	COP
<i>EE</i>	$G(T_h, T_c)$	$1 - \sqrt{\theta}$
<i>RR</i>	$G(T_h, T_c)$	$\frac{\sqrt{\theta}}{1 - \sqrt{\theta}}$
<i>PP</i>	$G(T_h, T_c)$	$\frac{1}{1 - \sqrt{\theta}}$
<i>PR</i>	$H(T_h, T_c)$	$\frac{1 + \theta}{1 - \theta}$
<i>RP</i>	$A(T_h, T_c)$	$\frac{1 + \theta}{1 - \theta}$

**Table 1.** Means  $T_i$  where  $T_h > T_i > T_c$  for various configurations, as explained in the text. COPs are given in terms of  $\theta = T_c/T_h$ .

Let us now consider another reservoir (warm) with temperature labeled as  $T_i$  such that  $T_h > T_i > T_c$ . Suppose, we are allowed to run a Carnot engine between reservoirs at  $T_h$  and  $T_i$ , or, between  $T_i$  and  $T_c$ . The respective efficiencies are :  $E_{hi} = (T_h - T_i)/T_h$  and  $E_{ic} = (T_i - T_c)/T_i$ . We can ask: for what value of  $T_i$ , do we have the situation:  $E_{hi} = E_{ic}$ ? It follows that at  $T_i = \sqrt{T_h T_c}$ , the two efficiencies are equal. Interestingly, the answer is invariant if we replace both the engines by two Carnot refrigerators or pumps, and require the equality of their respective COPs. In other words, if we set  $R_{hi} = R_{ic}$ , or  $P_{hi} = P_{ic}$ , the solution  $T_i$  is equal to the geometric mean temperature of  $T_h$  and  $T_c$ . We address these cases as *EE*, *RR*, and *PP*, respectively.

Further, we may consider a refrigerator between  $T_h$  and  $T_i$ , and a pump between  $T_i$  and  $T_c$ , and require that  $R_{hi} = P_{ic}$ . This



condition yields,  $T_i = (T_h + T_c)/2 = A(T_h, T_c)$ . Conversely, for the situation when  $P_{hi} = R_{ic}$ , we get  $T_i = H(T_h, T_c)$ . The latter cases are addressed as *RP* and *PR*, respectively. Thus, we note an interesting instance of Pythagorean means in the context of Carnot machines. These cases are depicted in *Figure 1*. See also *Table 1*.

### 3. Generalizations

Consider a set of  $k$  heat reservoirs with temperatures ordered as follows:

$$T_1 > T_2 > \dots > T_{k-1} > T_k. \tag{7}$$

Suppose we run a Carnot engine between a pair of reservoirs, one being a hot reservoir at  $\{T_i | i = 1, 2, \dots, k - 1\}$  and the other being the coldest reservoir within the set, at  $T_k$ . The efficiency of a particular engine would be  $E_{ik} = (T_i - T_k)/T_i$ . Then the average efficiency of all these engines may be defined as the arithmetic mean of the individual efficiencies,  $\bar{E} = \sum_i E_{ik}/(k - 1)$ . It is easy to verify that:

$$\bar{E} = \frac{\bar{T} - T_k}{T} \tag{8}$$

where  $\bar{T} = H(T_1, T_2, \dots, T_{k-1})$ . Thus one can state this result as:

(a) *The arithmetic mean of the efficiencies  $\{E_{ik}\}$  between  $\{T_i\}$  and  $T_k$ , where  $i = 1, 2, \dots, k - 1$ , is equal to the efficiency of a Carnot engine between  $\bar{T}$  and  $T_k$ , where  $\bar{T}$  is harmonic mean of the  $\{T_i\}$ .*

We can make similar statements, when the machines in case are refrigerators or pumps, connecting a certain mean COP of the machine, to a certain mean over the hot reservoir temperatures. These statements are given below. The said connection is also shown in *Table 2*.

(b) *The harmonic mean of  $\{R_{ik}\}$  between  $T_i$  and  $T_k$ , where  $i = 1, 2, \dots, k - 1$  is equal to the R-coefficient of a refrigerator between  $T_A$  and  $T_k$ , where  $T_A$  is the arithmetic mean of  $\{T_i\}$ .*



Case	Machine	Mean COP	Mean $\{T_i\}$
(a)	Engine	$A(\{E_{ik}\})$	$H$
(b)	Refrigerator	$H(\{R_{ik}\})$	$A$
(c)	Pump	$H(\{P_{ik}\})$	$H$
(a')	Engine	$A(\{E_{1i}\})$	$A$
(b')	Refrigerator	$H(\{R_{1i}\})$	$H$
(c')	Pump	$H(\{P_{1i}\})$	$A$

**Table 2.** Means in the presence of multiple reservoirs with temperatures  $T_1 > T_2 > \dots > T_{k-1} > T_k$ . The table shows type of the mean of COPs versus the mean of the reservoir temperatures  $\{T_i\}$ , where  $i = 1, 2, \dots, k - 1$  for cases (a), (b) and (c), while  $i = 2, 3, \dots, k$  for cases (a'), (b') and (c').

**Proof:** We can write

$$\sum_{i=1}^{k-1} R_{ik}^{-1} = \sum_{i=1}^{k-1} \frac{T_i - T_k}{T_k} \tag{9}$$

which can be expressed as:

$$\frac{1}{k-1} \sum_{i=1}^{k-1} R_{ik}^{-1} = \frac{\sum_i T_i / (k-1) - T_k}{T_k} \tag{10}$$

or,

$$R_H^{-1} = \frac{T_A - T_k}{T_k}, \tag{11}$$

where,  $R_H$  is the harmonic mean of  $\{R_{ik}\}$ , and RHS is the inverse of the  $R$ -coefficient ( $R_{Ak}$ ) between  $T_A$  and  $T_k$ .

(c) *The harmonic mean of  $\{P_{ik}\}$  between  $\{T_i\}$  and  $T_k$ , where  $i = 1, 2, \dots, k - 1$ , is equal to the  $P$ -coefficient of a heat pump between  $T_H$  and  $T_k$ , where  $T_H$  is the harmonic mean of  $\{T_i\}$ .*

Alternately, we can assume individual operations of the machines between the hottest reservoir  $T_1$  and each of the lower-temperature reservoirs. Then we can formulate statements, equivalent to the above and summarized below, that connect the means over the COPs to certain means over the lower temperatures.

(a') *The arithmetic mean of efficiencies  $\{E_{1i}\}$  between  $T_1$  and  $\{T_i\}$ , where  $i = 2, \dots, k$  is equal to the efficiency of an engine between  $T_1$  and the arithmetic mean of  $\{T_i\}$ .*

(b') *The harmonic mean of  $\{R_{1i}\}$  between  $T_1$  and  $\{T_i\}$ , where  $i = 2, \dots, k$ , is equal to the  $R$ -coefficient of a refrigerator between  $T_1$  and the harmonic mean of  $\{T_i\}$ .*



(c') The harmonic mean of  $\{P_{1i}\}$  between  $T_1$  and  $\{T_i\}$ , where  $i = 2, \dots, k$ , is equal to the  $P$ -coefficient of a pump between  $T_1$  and the arithmetic mean of  $\{T_i\}$ s.

Now, consider that for the ordered set of temperatures, we are given the probability distribution  $\{p_i\}$ , to connect two temperatures  $T_i$  and  $T_k$  by a Carnot machine, say, a heat engine. Then we have :

$$\sum_i p_i E_{ik} = 1 - T_k \sum_i \frac{p_i}{T_i} \tag{12}$$

Here,  $\sum_i p_i/T_i = 1/\bar{T}$  defines the weighted harmonic mean of  $T_i$ s. Then the statement (a) in the above, is generalized as follows:

*For a given probability distribution  $\{p_i\}$ , the weighted arithmetic mean of the efficiencies  $\{E_{ik}\}$  between  $T_i$  and  $T_k$ , is equal to the efficiency of an engine between  $\bar{T}$  and  $T_k$ , where  $\bar{T}$  is the weighted harmonic mean of  $\{T_i\}$ ,  $i = 1, 2, \dots, k - 1$ .*

Similar generalized statements, based on the weighted means, can be made for the other cases mentioned above.

Finally, we consider an ordered set of  $n$  positive, real numbers (temperatures):

$$T_1 > T_2 > \dots > T_{k-1} > T_k > T_{k+1} > \dots > T_n. \tag{13}$$

We ask, under what condition, a temperature, say  $T_k$ , is one of the Pythagorean means? The solution generalizes the two-temperatures case, as discussed in Section 2. So, if  $T_k$  is the arithmetic mean of the rest of the values within the set, *i.e.*,  $T_k = A(T_1, T_2, \dots, T_{k-1}, T_{k+1}, \dots, T_n)$ , then the following relation can be written:

$$\sum_{i=1}^{k-1} R_{ik}^{-1} = \sum_{i=k+1}^n P_{ki}^{-1}. \tag{14}$$

In other words, the  $R$  coefficient enters in the above, for temperatures above  $T_k$  and the  $P$  coefficient, for temperatures below  $T_k$ .

Now from the definition of the harmonic mean, we have:

$$\sum_{i=1}^{k-1} R_{ik}^{-1} = (k - 1)R_H^{-1}, \tag{15}$$





and

$$\sum_{i=k+1}^n P_{ki}^{-1} = (n - k)P_H^{-1}, \tag{16}$$

where  $P_H$  is the harmonic mean of  $\{P_{ki}\}$ . Using (15) and (16) in (14), we get:

$$\frac{R_H}{P_H} = \frac{k - 1}{n - k}. \tag{17}$$

It implies that the ratio of effective  $R$ - to  $P$ - coefficients which appear as harmonic means, is the ratio of whole numbers. This results is somewhat reminiscent of the original observation of Pythagoras about the ratios of means, expressible as ratios of whole numbers.

Similarly, we can consider the occurrence of other means within the given set of real, positive numbers. It is left as an exercise for the reader that if  $T_k = H(T_1, T_2, \dots, T_{k-1}, T_{k+1}, \dots, T_n)$ , then the following condition is satisfied:

$$\sum_{i=1}^{k-1} P_{ik}^{-1} = \sum_{i=k+1}^n R_{ki}^{-1}. \tag{18}$$

Finally, if  $T_k = G(T_1, T_2, \dots, T_{k-1}, T_{k+1}, \dots, T_n)$ , for some given  $k$ , then we can have three situations for a pair of temperatures  $(T_i, T_k)$ , with  $i \neq k$ : all  $E$  coefficients, all  $R$  coefficients, or all  $P$  coefficients. Then it can be shown that:

$$\prod_{i=1}^{k-1} (1 - E_{ik}) = \prod_{i=k+1}^n (1 - E_{ki}), \tag{19}$$

$$\prod_{i=1}^{k-1} (1 + R_{ik}^{-1}) = \prod_{i=k+1}^n (1 + R_{ki}^{-1}), \tag{20}$$

$$\prod_{i=1}^{k-1} (P_{ik}^{-1} - 1) = \prod_{i=k+1}^n (P_{ki}^{-1} - 1). \tag{21}$$

#### 4. Summary

Concluding, we have highlighted a few instances where the Pythagorean means arise naturally in the discussion of coefficients of perfor-

The ratio of effective  $R$ - to  $P$ - coefficients which appear as harmonic means, is the ratio of whole numbers.



mance for Carnot machines working between hot and cold reservoirs. In comparing the COPs of a machine between  $T_h$  and  $T_i$ , and of a machine between  $T_i$  and  $T_c$ , we have considered scenarios like  $EE, RR, PP, RP$ , and  $PR$ . So far, we did not consider an engine along with a refrigerator or a pump. The intuition behind this was to compare the performance of similar machines. Thus, while a refrigerator or a heat pump drives heat from cold to hot, in a heat engine the heat flow is in opposite direction. Further, it is not possible to consider a case like  $EP$ , because while  $E < 1$ , we have  $P > 1$ , and so their COPs cannot become equal. But scenario like  $ER$  or  $RE$  are possible. It can be shown that the equality of respective COPs in these cases, is possible only if the condition  $\theta < 3 - 2\sqrt{2}$  holds. Then it leads to a different kind of symmetric mean, and not to a Pythagorean mean.

We also considered various physical situations involving more than one reservoirs ( $T_i$ ) that can be coupled *via* a Carnot machine to a given reservoir ( $T_k$ ), and the mean COP has been related to the effective COP evaluated from the mean temperature and  $T_k$ . The relations pointed out in this paper and the known inequalities between Pythagorean means may be useful to fix bounds on the performance of thermal machines in such physical situations. For instance, consider the fact that the harmonic mean is dominated by the lowest number:  $H(\{R_{ik}\}) \leq (k - 1)\text{Min}(\{R_{ik}\})$ . From the definition,  $R_{ik} = T_k/(T_i - T_k)$ , we have  $\text{Min}(\{R_{ik}\}) = R_{1k}$ , since  $T_1$  is the largest entry. Then from (15), we get the upper bound for the effective COP,  $R_{Ak} \leq (k - 1)R_{1k}$ .

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