Vibrations and Eigenvalues

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The vibrating string problem is the source of much mathematics and physics. This article describes Lagrange's formulation of a discretised version of the problem and its solution. This is also the first instance of an eigenvalue problem.

Vibrations occur everywhere. My speech reaches you by a series of vibrations starting from my vocal chords and ending at your ear drums. We make music by causing strings, membranes, or air columns to vibrate. Engineers design safe structures by controlling vibrations.

I will describe to you a very simple vibrating system and the mathematics needed to analyse it. The ideas were born in the work of Joseph-Louis Lagrange (1736–1813), and I begin by quoting from the preface of his great book *Méchanique Analitique* published in 1788:

We already have various treatises on mechanics but the plan of this one is entirely new. I have set myself the problem of reducing this science [mechanics],and the art of solving the problems pertaining to it, to general formulae whose simple development gives all the equations necessary for the solution of each problem ... No diagrams will be found in this work. The methods which I expound in it demand neither constructions nor geometrical or mechanical reasonings, but solely algebraic [analytic] operations subjected to a uniform and regular procedure. Those who like analysis will be pleased to see mechanics become a new branch of it, and will be obliged to me for having extended its domain.

Keywords
Vibrations, eigenvalues, tridiagonal matrix.

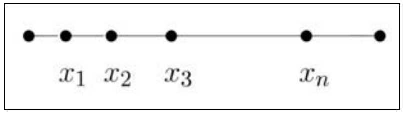

Figure 1.

Consider a long thin tight elastic string (like the wire of a *veena*) with fixed end points. If it is plucked slightly and released, the string vibrates. The problem is to find equations that describe these vibrations and to find solutions of these equations. The equations were first found by Jean d'Alembert, and two different forms of the solution were given by him and by Leonhard Euler.

Lagrange followed a different path: he discretised the problem. Imagine the string is of length $(n + 1)d$, has negligible mass, and there are n beads of mass m each placed along the string at regular intervals d (*Figure* 1):

The string is pulled slightly in the y-direction and the beads are displaced to positions $y_1, y_2, ..., y_n$ (*Figure* 2). The tension T in the string is a force that pulls the beads towards the initial position of rest. Let α be the angle that the string between the $(j - 1)$th and the jth bead makes with the x-axis.

If α is small, then $\sin\alpha$ is close to $\tan\alpha$.

Then the component of T in the downward direction is $T\ \sin\alpha$ (*Figure* 3). If α is small, then $\cos\alpha$ is close to 1, and $\sin\alpha$ is close to $\tan\alpha$. Thus the downward component of T is approximately:

$$T\tan\alpha = T\frac{y_j - y_{j-1}}{d}.$$

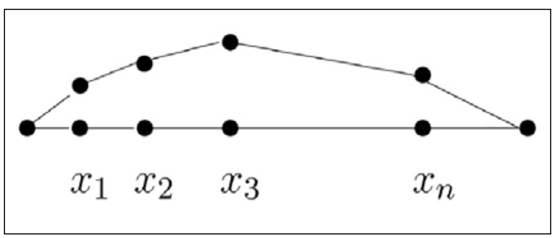

Figure 2.

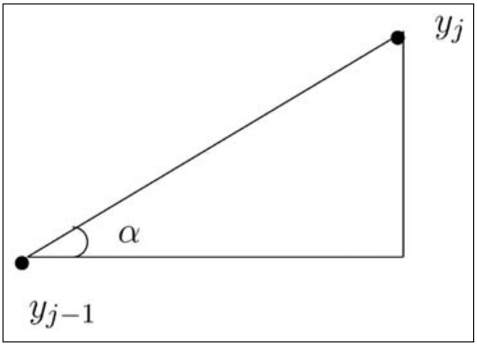

Figure 3.

Similarly, the pull exerted on the jth bead from the other side of the string is:

$$T\frac{y_j - y_{j+1}}{d}.$$

Thus, the total force exerted on the jth bead is:

$$\frac{T}{d}(2y_j - y_{j-1} - y_{j+1}).$$

By Newton's second law of motion,

$$\text{Force} = \text{mass} \times \text{acceleration},$$

this force is equal to $m\ddot{y}_j$, where the two dots denote the second derivative with respect to time. So we have,

$$m\ddot{y}_j = \frac{-T}{d}(2y_j - y_{j-1} - y_{j+1}). \quad (1)$$

A simple application of Newton's second law gives the equations governing the motion of the beads.

The minus sign outside the brackets indicates that the force is in the 'downward' direction. We have n equations, one for each $1 \leq j \leq n$. It is convenient to write them as a single vector equation:

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_n \end{bmatrix} = \frac{-T}{md} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & & \\ & & & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad (2)$$

The problem is reduced to that of finding the eigenvalues and eigenvectors of a simple tridiagonal matrix L. This L is *the* most important matrix in applied mathematics.

or as

$$\ddot{\boldsymbol{y}} = \frac{-T}{md} L\boldsymbol{y}, \tag{3}$$

where $\boldsymbol{y}$ is the vector with n components $y_1, y_2, \ldots, y_n$ and L is the $n \times n$ matrix with entries $l_{ii} = 2$ for all i, $l_{ij} = -1$ if $|i - j| = 1$, and $l_{ij} = 0$ if $|i - j| > 1$. (A matrix of this special form is called a 'tridiagonal' matrix.)

Let us drop the factor $-T/md$ (which we can reinstate later) and study the equation

$$\ddot{\boldsymbol{y}} = L\boldsymbol{y} \,. \tag{4}$$

We want to find solutions of this equation; *i.e.*, we want to find $\boldsymbol{y}(t)$ that satisfy (4). In this, we are guided by two considerations. Our experience tells us that the motion of the string is oscillatory; the simplest oscillatory function we know of is $\sin t$, and its second derivative is equal to itself with a negative sign. Thus it would be reasonable to think of a solution:

$$\boldsymbol{y}(t) = (\sin \omega t)\boldsymbol{u} \,. \tag{5}$$

If we plug this into (4), we get,

$$-\omega^2 (\sin \omega t)\boldsymbol{u} = (\sin \omega t) L\boldsymbol{u} \,.$$

So, we must have

$$L\boldsymbol{u} = -\omega^2 \boldsymbol{u} \,.$$

In other words, $\boldsymbol{u}$ is an 'eigenvector' of L corresponding to 'eigenvalue' $-\omega^2$.

So our problem has been reduced to a problem on matrices: find the eigenvalues and eigenvectors of the tridiagonal matrix L. In general, it is not easy to find eigenvalues of a (tridiagonal) matrix. But our L is rather special. The calculation that follows now is very ingenious, and remarkable in its simplicity.

The characteristic equtaion $L\boldsymbol{u} = \lambda \boldsymbol{u}$ can be written out as:

$$-u_{j-1} + 2u_j - u_{j+1} = \lambda u_j, 1 \leq j \leq n, \tag{6}$$

together with the boundary conditions,

$$u_0 = u_{n+1} = 0. \tag{7}$$

The calculation leading to the eigenvalues and eigenvectors of L is ingenious and remarkable in its simplicity.

The two conditions in (7) stem from the fact that the first and the last row of the matrix L are different from the rest of the rows. This is because the two endpoints of the string remain fixed – their displacement in the y-direction is zero. The trigonometric identity,

$$\begin{aligned}\sin(j+1)\alpha + \sin(j-1)\alpha &= 2\sin j\alpha\cos\alpha \\ &= 2\sin j\alpha\left(1 - 2\sin^2\frac{\alpha}{2}\right),\end{aligned}$$

after a rearrangement, can be written as:

$$-\sin(j-1)\alpha + 2\sin j\alpha - \sin(j+1)\alpha = \left(4\sin^2\frac{\alpha}{2}\right)\sin j\alpha. \tag{8}$$

So, the equations (6) are satisfied if we choose

$$\lambda = 4\sin^2\frac{\alpha}{2}, \quad u_j = \sin j\alpha. \tag{9}$$

There are some restrictions on α. The vector $\boldsymbol{u}$ is not zero and hence α cannot be an integral multiple of π. The first condition in (7) is automatically satisfied, and the second dictates that $\sin(n+1)\alpha = 0$.

This, in turn means that $\alpha = k\pi/(n+1)$. Thus the n eigenvalues of L are:

$$\lambda = 4\sin^2\frac{k\pi}{2(n+1)}, \ k = 1, 2, \ldots, n. \tag{10}$$

You can write out for yourself the corresponding eigenvectors.

What does this tell us about our original problem? You are invited to go back to ω and to (3) and think. A bit of 'dimension analysis' is helpful here. The quantity T in (3) represents a force. So its units are $\frac{\text{mass}\times\text{length}}{(\text{time})^2}$. The units of $\frac{T}{md}$ are, therefore $(\text{time})^{-2}$. So, after the factor $\frac{-T}{md}$ is reinstated , the quantity ω represents a frequency. This is the frequency of oscillation of the string. It is proportional to $\sqrt{T/md}$. So, it increases with the tension and decreases with the mass m of the beads and the distance d between them. Does this correspond to your physical experience?

We can go in several directions from here. Letting d go to zero we approach the usual string with uniformly distributed mass. The matrix L then becomes a differential operator. The equation corresponding to (3) then becomes Euler's equation for the vibrating string. We can study the problem of beads on a heavy string. Somewhat surprising may be the fact that the same equations describe the flow of electricity in telephone networks.

The study of the vibrating string led to the discovery of 'Fourier series', a subject that eventually became 'harmonic analysis', and is behind much of modern technology – from CT scans to fast computers.

I end this talk by mentioning a few more things about Lagrange. Many ideas in mechanics go back to him. It has been common to talk of 'Lagrangian mechanics' and 'Hamiltonian mechanics' as the two viewpoints of this subject. Along with L Euler, he was the founder of the 'calculus of variations'. The problem that led Lagrange to this subject was his study of the 'tautochrone', the curve moving on which a weighted particle arrives at a fixed point in the same time independent of its initial position. The Lagrange method of undetermined multipliers is one of the most used tools for finding maxima and minima of functions of several variables. Every student of group theory learns Lagrange's theorem that the order of a subgroup H of a finite group G divides the order of G. In number theory, he proved several theorems, one of which called 'Wilson's theorem' says that n is a prime if and only if $(n-1)!+1$ is divisible by n. In addition to all this work, Lagrange was a member of the committee appointed by the French Academy of Sciences to standardise weights and measures. The metric system with a decimal base was introduced by this committee.

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Suggested Reading

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