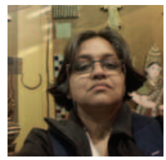
# Gravity Defied
# From Potato Asteroids to Magnetised Neutron Stars

## 3. White Dwarfs (Dead Stars of the First Kind)

*Sushan Konar*

**During its active lifetime, a star burns its nuclear fuel, and gravitation is held off by the pressure of the heated gas. Gravity should take over once this fuel is exhausted unless some other agency saves the star from such a fate. Low mass stars find peace as 'white dwarfs' when the electrons settle into a Fermi degenerate phase where the pressure of degenerate electrons balance the gravitational pressure.**



**Sushan Konar works at NCRA-TIFR, Pune. She tries to understand the physics of stellar compact objects (white dwarfs, neutron stars) for her livelihood and writes a blog about life in academia as a hobby.**

## 1. The Stars

The nature of the stars has been questioned and debated over ever since the dawn of human intelligence. Yet, it's only in the late nineteenth century when the Sun and other stars have been understood to be self-gravitating gaseous objects. We now know that the Sun is powered by nuclear fusion, producing helium from hydrogen, the direct evidence of which has come from the detection of solar neutrinos in 1968. In fact, it is this particular nuclear reaction that defines the birth of a star. When a self-gravitating gas cloud attains the capability of fusing hydrogen into helium, a star is born!

Giant molecular clouds in the interstellar medium are the birthplaces of new stars (*Figure* 1). The interstellar medium consists mainly of atomic hydrogen ($\rho \sim 0.1$ atoms/cm$^3$, $T \sim 10^4$ K). This exceptionally diffuse gas hosts two types of massive clouds ($10^7$ M$_\odot$) of denser gas of which the more dense ones consist almost entirely of molecular hydrogen. Supernova blast waves, impacting these molecular clouds, create shock waves, heating the clouds up to $T \sim 10^6$ K (temperature required for hydrogen

**Keywords**
Nuclear fusion, degeneracy pressure, relativistic effects, TOV equation.

**Figure 1.** Pillars of molecular clouds in Eagle nebula – a birthplace of stars. (Image courtesy – `https://jwst.nasa.gov/`)

fusion) which ultimately trigger star formation.

For a given temperature, the gas would be stable for sufficiently small masses but would begin a process of runaway contraction, when a critical mass is exceeded.

However, before actual star formation 'Jeans instability' causes the collapse and subsequent fragmentation of these molecular clouds into star-sized clumps. We have seen earlier, how the condition for hydrostatic equilibrium depends on the temperature ($T$) (responsible for the gas pressure) and the mass ($M$) of the gas[1]. The gas cloud collapses if its temperature is not sufficient to balance the gravitational pressure. Conversely, for a given temperature, the gas would be stable for sufficiently small masses but would begin a process of runaway contraction, when a critical mass is exceeded.

[1] S Konar, Gravity Defied: From Potato Asteroids to Magnetised Neutron Stars; 1: The Self-Gravitating Objects, *Resonance*, Vol.22, No.3, pp.225–235, 2017.

This critical mass, known as the 'Jeans mass' (after Sir James Jeans) can be estimated as follows. Consider an isothermal gravitational contraction of a homogeneous, spherical gas cloud of mass $M$, and radius $r$ which is at a temperature $T$. When the gravitational energy release due to contraction, exceeds the work done on the gas cloud, we have an episode of 'runaway contraction'. Therefore, the critical mass is obtained when the work done equals the gravitational energy release. The gravitational energy release from the gas cloud when it contracts from a radius $r$ to

$r - dr$ is,

$$dE_G = G\frac{M^2}{r^2}dr\,, \qquad (1)$$

and the work done on the gas cloud during this process is,

$$dW \propto \rho T r^2 dr\,, \qquad (2)$$

where $\rho$ is the density of the gas and the cloud is assumed to behave like an idea gas ($P \sim \rho T$). Equating $dE_G$ and $dW$ we arrive at the Jeans mass given by,

$$M_J \propto \left(\frac{T^3}{n}\right)^{\frac{1}{2}}\,. \qquad (3)$$

It should be noted here that the above formulation assumes an isothermal contraction. For adiabatic processes, the Jeans mass turns out to be,

$$M_J^a \propto \rho^{\frac{3}{2}(\gamma-\frac{4}{3})}\,, \qquad (4)$$

where $\gamma$ is the adiabatic index of the gas. It can be immediately seen that $M_J$ increases with increasing density for $\gamma > 4/3$, and decreases with increasing density for $\gamma < 4/3$.

In summary, then the path taken by a giant molecular cloud to star formation is as follows:

- Giant molecular clouds of gas and dust begin with a mass of $\sim 10^3 - -10^6$ $M_\odot$ and an initial composition of hydrogen with a small mixture of helium, molecular hydrogen, water, and silicates.
- The clouds collapse gravitationally and fragment into clumps. The small clouds stick together and grow through accretion, and are compressed by supernova blast waves.
- The clumps collapse and fragment further to form protostars.
- Protostars collapses till hydrogen fusion starts at the core.

For most of a star's life the gravitational force and the gas pressure balance each other. This balance is finely-tuned and self-regulating: if the rate of energy generation in the core slows down, gravity wins out over pressure, and the star begins to contract. This contraction increases the temperature and pressure of the stellar interior, which leads to higher energy generation rates and a return to equilibrium.

Once formed, the stars spend most of their life fusing hydrogen, in the phase known as 'main sequence'. During main sequence, the stars get progressively hotter and brighter (higher luminosity). Depending on the mass, the stars then enter into subsequent phases of nuclear fusion involving helium and elements upwards of helium. Fusion is an exothermic process that generates energy. During this process, when two nuclei of lighter mass fuse into a heavier nucleus, the mass per nucleon actually decreases, and this mass defect shows up as energy. The mass per nucleon keeps decreasing with increasing atomic number till $Fe^{56}$ (*Figure* 2). Beyond this point, mass per nucleon increases and fusion can only be achieved by supplying energy from outside. Therefore, the nuclear source of fuel is completely exhausted upon the production of $Fe^{56}$.

Therefore, the ultimate phase of stable equilibrium of a star depends on its initial mass. In this phase, gravitation is stably balanced by electromagnetic or quantum effects and there is no evolution hereafter. The evolution and the end-states are described in *Figure* 3 and *Table* 1.

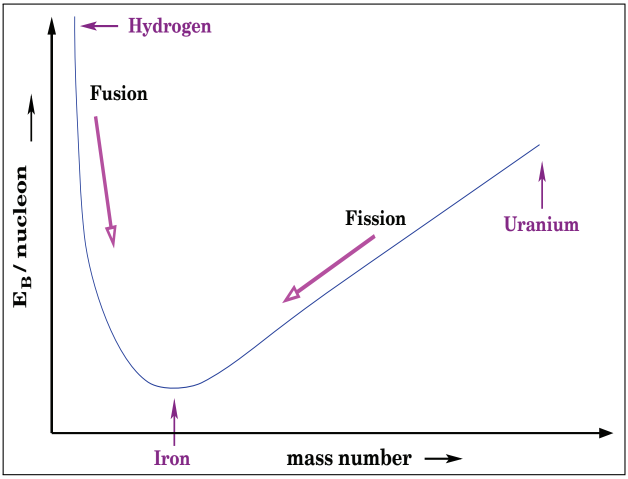



**Figure 2.** Binding energy ($E_B$) per nucleon in stable atomic nuclei. $Fe^{56}$ has the minimum binding energy per nucleon. Therefore, beyond $Fe^{56}$, fusion of elements no longer release energy. Stellar fuel is exhausted once $Fe^{56}$ is made.

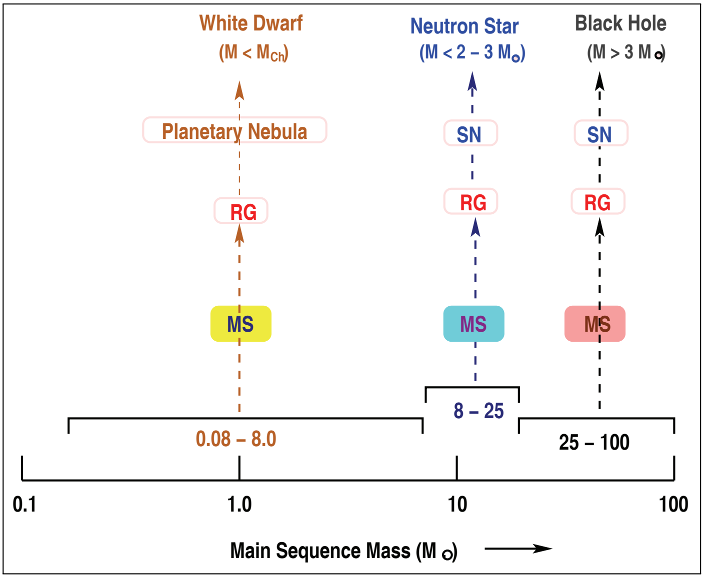



**Figure 3.** The evolutionary path and the final remnant (white dwarf / neutron star / black hole) of a star depends crucially on its initial mass. (Legends: MS – main sequence, RG – red giant, SN – supernova

## 2. White Dwarfs

The equations governing the structure of an active star are:

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \quad \text{hydrostatic equilibrium;} \tag{5}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r), \quad \text{mass-radius relation;} \tag{6}$$

$$P = P(\rho, T), \quad \text{equation of state;} \tag{7}$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho[\epsilon_n(r) - \epsilon_\nu(r)], \quad \text{energy generation;} \tag{8}$$

$$\frac{dT(r)}{dr} \propto -\frac{\rho(r)}{T(r)^3}\frac{L(r)}{4\pi r^2}, \quad \text{energy transport ;} \tag{9}$$

where $T$ is the temperature and $L$, $\epsilon_n$, $\epsilon_\nu$ are the total, the nuclear, and the neutrino luminosities respectively. The active phase ends when a star runs out of its nuclear fuel – either because the star never reaches the temperature required to fuse the next element (low mass stars), or because it reaches the end of the fusion chain by producing $Fe^{56}$ (for stars with $M > 8 - 10\ M_\odot$). When low mass stars stop burning their nuclear fuel, they typically go through a red-giant and a planetary nebula phase, and ends up with no further energy generation. As the residual heat is radiated away, it cools down and reaches a temperature low enough for it to go into a quantum degenerate phase (*Figure* 4). For this

The active phase ends when a star runs out of its nuclear fuel – either because the star never reaches the temperature required to fuse the next element, or because it reaches the end of the fusion chain.

**Table 1.** The end state of a star as a function of its mass at the beginning of the main-sequence ($H \rightarrow He^4$) phase. (Legend : WD – white dwarf)

| **Main-Sequence Mass** | **Final State** | ***Gravity-Resisting Agent*** |
|---|---|---|
| $\lesssim 0.01\,\mathrm{M}_\odot$ | Planet | van der Waals |
| $0.01 \lesssim M/\mathrm{M}_\odot \lesssim 0.08$ | Brown Dwarf | Fermi degeneracy |
| $0.08 \lesssim M/\mathrm{M}_\odot \lesssim 0.5$ | He WD | - do - |
| $0.5 \lesssim M/\mathrm{M}_\odot \lesssim 8$ | C-O WD | - do - |
| $8 \lesssim M/\mathrm{M}_\odot \lesssim 10$ | O-Ne-Mg WD | - do - |
| $10 \lesssim M/\mathrm{M}_\odot \lesssim 25-40$ | Neutron Star | - do - |
| $40 \lesssim M/\mathrm{M}_\odot$ | Black Hole | none |

stellar remnant, the structure is governed only by the first three equations above and the most important ingredient is the equation of state of the degenerate matter.

*Table* 1 shows stars with different initial masses ending up as white dwarfs with different compositions. For all of them though, the pressure comes from the degenerate electrons. However, it needs to be remembered that with increasing density, the Fermi energy of the electrons increase. When the density reaches $\sim 10^6\ \mathrm{g\,cm^{-3}}$ the electron momenta are high enough for them to be treated as relativistic. In that case, the relation between the energy and the momentum changes from $E = p^2/2m$ for non-relativistic particles to $E = \sqrt{p^2c^2 + m^2c^4}$ for the relativistic case. Consequently, the pressure is now given by,

$$P = \frac{8\pi m^4 c^5}{3h^3}\Phi(x)\,, \tag{10}$$

**Figure 4.** NGC 2440, a planetary nebula, contains one of the hottest white dwarfs (seen as a bright dot near the centre). (Image courtesy: `https://www.nasa.gov/`)

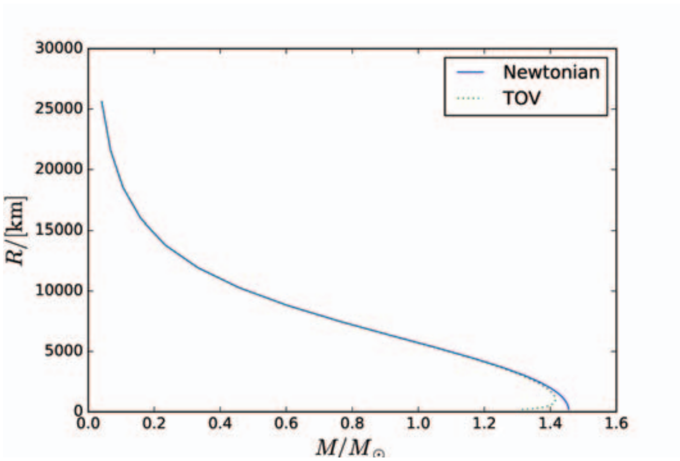



**Figure 5.** Mass-Radius relation of purely electron-degenerate white dwarfs. The dotted curve is for calculations that are inclusive of general relativistic corrections. (Picture courtesy – Prasanta Bera, IUCAA, Pune)

where $x = p_F/mc^2$, $p_F$ being the Fermi momentum. The function $\Phi(x)$ is given by,

$$\Phi(x) = \frac{1}{8\pi^2}\left(x\sqrt{1+x^2}\,(2x^2/3-1) + \ln\left[x + \sqrt{1+x^2}\right]\right). \quad (11)$$

Using this equation of state, the structure and the mass-radius relation of white dwarfs is calculated, as has been shown in *Figure* 5.

The most interesting consequence of the above equation of state is the case of ultra-relativistic electrons. In the limit, $p_F \to mc$ we have $P \propto \rho^{4/3}$. Following the arguments, of section 1.2.2 of the previous article in this series[2], we arrive at the startling result of,

$$M = \underline{\text{constant}}\,, \quad (12)$$

which is indicated by the mass-radius curve meeting the mass-axis at zero radius. The meaning of this is the existence of an upper limit for the mass of a self-gravitating body supported purely by electron degeneracy pressure. This indeed is the famous Chandrasekhar mass limit ($M_{Ch} \simeq 1.44 M_\odot$) for white dwarfs. In 1925, a star as massive as the Sun but with radius similar to the Earth was discovered. Following R H Fowler's suggestion that such a star has densities in which quantum effects would be important, S Chandrasekhar, then a student at Presidency College, Madras, developed the theory of white dwarfs.

[2] S Konar, Gravity Defied: From Potato Asteroids to Magnetised Neutron Stars, 2: The Failed Stars, *Resonance*, Vol.22, No.4, pp.389–398, 2017.

### 2.1 *New Directions*

With increasing compactness, the effects of general relativity becomes important and the simple hydrostatic equilibrium equation used to calculate the structure of a self-gravitating object turns out to be inadequate.

Theoretical research into the interior properties of white dwarfs has received a boost in recent times. In particular, two physical effects have been investigated in detail in the context of its structure. Here we take a look at these new developments.

**2.1.1** ***Relativistic Effects*****:** We have seen earlier that white dwarfs, made up of completely non-relativistic degenerate electrons, obey the following mass-radius relation,

$$MR^3 = \text{constant}\,, \tag{13}$$

implying that with increasing mass, a white dwarf gets more and more compact. Though, the mass-radius relation changes from such a simple one (as seen in *Figure* 5) as the constituent electrons become relativistic, the essential behaviour remains the same. With increasing compactness, the effects of general relativity becomes important and the simple hydrostatic equilibrium equation used to calculate the structure of a self-gravitating object turns out to be inadequate.

The hydrostatic equilibrium equation, modified to incorporate the general relativistic effects, goes by the name of 'Tolman–Oppenheimer–Volkoff' (TOV) equation after the scientists who derived it for the first time and is given by the following form:

$$\frac{dP}{dr} = -\frac{G\left(M(r) + 4\pi r^3 P(r)/c^2\right)\left(\rho(r) + P(r)/c^2\right)}{r^2 - 2GM(r)r/c^2}, \tag{14}$$

for the structure of a static, spherically symmetric, relativistic star in isotropic coordinates. To derive this equation from the first principles is beyond the scope of the present article. However, we can get a sense of the modifications by realising that in general, relativity, mass, and energy are equivalent quantities and so are the mass density and energy density (pressure). Modification terms in TOV, compared to the simple, non-relativistic hydrostatic equilibrium equation, arise basically from this concept.

The structure of a relativistic star is typically calculated using the above TOV equation. However, there is an easy way to find out the significance of the relativistic effects on a given star. This can be done by quantifying the compactness in the following way. Using (13) we find that the escape velocity from the equatorial surface of a white dwarf is,

$$V_{\rm E} = (2GM/R)^{1/2} \propto M^{2/3} \,. \quad (15)$$

In a self-gravitating object the currents that generate this magnetic field necessarily flow somewhere inside that object.

A comparison of this escape velocity with the velocity of light, gives us a measure of the compactness of the object. For a typical white dwarf of $M \simeq 1{\rm M}_\odot$ and $R \simeq 10^4$ KM, the compactness parameter turns out to be,

$$\frac{V_{\rm E}}{c} \simeq (2GM/R)^{1/2} \sim 0.02 \,, \quad (16)$$

indicating that the effect of relativistic corrections to the structure of white dwarfs are not very important. This can be readily seen from *Figure* 5. The dotted curve, giving the mass-relativistic relation obtained from the TOV equation, deviates only a little from the mass-relativistic relation obtained from non-relativistic hydrostatic equation.

**2.1.2** ***Magnetic Fields*****:** It is well known that magnetic field is omnipresent in the Universe. In a self-gravitating object the currents that generate this magnetic field necessarily flow somewhere inside that object (this is not about astrophysical objects in external fields!). Effectively, this gives rise to a $\mathbf{J} \times \mathbf{B}$ force ($\mathbf{J}$ – current density, $\mathbf{B}$ – magnetic field) which need to be balanced by a redistribution of matter. The structure calculation of a self-gravitating magnetised object therefore requires a modification of the hydrostatic equilibrium equation in the following way:

$$\frac{dP_{\rm g}}{dr} + \frac{dP_{\rm m}}{dr} = -\rho(r)g(r), \quad (17)$$

where $P_{\rm g}$ and $P_{\rm m}$ are the pressures exerted by the gas and the magnetic field.

The above formulation automatically constrains the maximum strength of the magnetic field for which a self-gravitating configuration would be stable. *Table* 2 shows the observed and the theoretical maximum value of the magnetic field for a number of astrophysical objects. It is obvious that except for white dwarfs and neutron stars, the observed value of the field is completely insignificant compared to the theoretical maximum and hence

**Table 2.** The observed and theoretical maximum values of the magnetic field for various astrophysical objects.

| | Mass | Radius | $B^{surface}_{average}$ | $B^{central}_{max}$ |
|---|---|---|---|---|
| | $M_\odot$ | cm | G | G |
| Earth | $3 \times 10^{-6}$ | $6 \times 10^{8}$ | $\lesssim 1$ | $2 \times 10^{7}$ |
| Jupiter | $10^{-3}$ | $7 \times 10^{9}$ | $\sim 10$ | $5 \times 10^{7}$ |
| Brown Dwarf | $0.01 - 0.1$ | $10^{10}$ | $1 - 10^{3}$ | $10^{7} - 10^{8}$ |
| Sun | 1.0 | $10^{11}$ | $\sim$ few | $3 \times 10^{8}$ |
| White Dwarf | 1.4 ($M_{Ch}$) | $10^{9}$ | $10^{3} - 10^{9}$ | $4 \times 10^{12}$ |
| Neutron Star | $\lesssim 2.0$ | $10^{6}$ | $10^{8} - 10^{15}$ | $5 \times 10^{18}$ |

plays no role in modifying the structure. On the other hand, much effort is being directed to understand the structure and nature of the magnetic fields in white dwarfs and neutron stars at present. A series of excellent papers by Prasanta Bera and Dipankar Bhattacharya address many such questions (incorporating general relativistic corrections and strong magnetic fields) related to the white dwarfs.

## Acknowledgement

Technological advancements led to the discovery of a large number of white dwarfs in recent times, spurring intense theoretical interest. As a result, a fortuitous opportunity to work with Rajaram Nityananda occurred, which allowed me to learn more about white dwafs beyond the mandatory graduate course material.

*Address for Correspondence*
Sushan Konar
National Centre for Radio Astrophysics
TIFR
Pune 411 007 India.
Email:
sushan@ncra.tifr.res.in

## Suggested Reading

**[1] G Srinivasan, *What are the Stars?*, Universities Press, Hyderabad, 2011.**
**[2] G Srinivasan, *Can Stars Find Peace?*, Universities Press, Hyderabad, 2011.**
**[3] S D Kawaler, White Dwarf Stars in *Stellar Remnants, Lecture Notes 1995*, Saas-Fee Advanced Course 25, Springer-Verlag, Berlin, 1997.**
**[4] S Chandrasekhar, The Highly Collapsed Configurations of a Stellar Mass, *MNRAS*, Vol.91, pp.456–466, 1931.**
**[5] S Chandrasekhar, The Maximum Mass of Ideal White Dwarfs, *ApJ*, Vol.74, pp.81–82, 1931.**