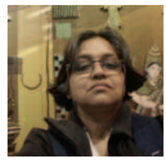
Gravity Defied
From Potato Asteroids to Magnetised Neutron Stars
2. The Failed Stars

Sushan Konar

Gravitation, the universal attractive force, acts upon all matter (and radiation) relentlessly. Stable extended structures can exist only when gravity is held off by other forces of nature. This series of articles explores this interplay, looking at objects that just missed being stars in this particular installment.



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1. Self-Gravitating Fluids

The first resistance that Nature puts up against gravitation is the rigidity of solids. We have seen how small rocky objects and terrestrial planets take recourse to this to retain their extended structures against gravitational pull [1]. The situation is entirely different in largely fluid (gas/liquid) Jovian planets. A fluid does not have rigidity. Instead, an element of fluid would be in hydrostatic equilibrium (at rest or at constant velocity) when external forces such as gravity are balanced by a pressure gradient. In other words, in a self-gravitating object, gravity pulling the material inwards is balanced by the internal pressure differential (due to heat or quantum effects) pushing the material outwards (*Figure* 1).

1.1 *Hydrostatic Equilibrium*

Consider the fluid element in a cylindrical region of length dr, area dA, at a distance r from the centre of a self-gravitating object having a density $\rho(r)$. Then, the volume of the cylinder is $dr\,dA$ and its mass $dm = \rho(r)\,dr\,dA$. The force of gravity acting on the

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Hydrostatic equilibrium, degeneracy pressure, Jovian planet, brown dwarf.

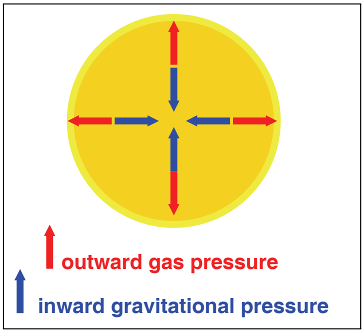

Figure 1. Balance of pressure in a self-gravitating fluid object.

cylinder is,

$$F_{\mathrm{G}} = -\frac{\mathrm{G}M(r)\,dm}{r^2} = -\frac{\mathrm{G}M(r)\,\rho(r)\,dr\,dA}{r^2}\,, \tag{1}$$

In a self-gravitating object, gravity pulling the material inwards is balanced by the internal pressure differential due to heat or quantum effects, pushing the material outwards.

where $M(r)$ is the total mass contained within the radius r. This gravitational force is balanced by the net difference in pressure, dP, from above and below (*Figure* 2). In equilibrium, the effective pressure force should be equal to the gravitational force giving us,

$$dP = -\frac{\mathrm{G}M(r)\,\rho(r)\,dr}{r^2} \;\Rightarrow\; \frac{dP}{dr} = -\frac{\mathrm{G}M(r)\,\rho(r)}{r^2}\,. \tag{2}$$

This equation of hydrostatic equilibrium governs the structure of all self-gravitating fluid bodies, including those of most stars. This simple form is, of course, modified when relativistic effects become important which we shall consider later.

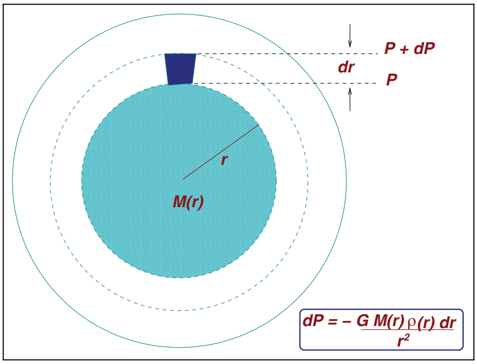

Figure 2. Equilibrium of a fluid element inside a self-gravitating object, defining the equation of hydrostatic balance of pressure.

1.2 *Gas Pressure*

1.2.1 ***Kelvin–Helmholtz Mechanism***: The Kelvin–Helmholtz mechanism can be thought of as a dynamic process through which a self-gravitating body attains successive phases of hydrostatic equilibrium. To begin with, a gas cloud contracts under the mutual gravitational attraction of its constituent particles. If a spherical gas cloud of mass M and initial radius R_1 contracts to a final radius of R_2, then the net change $\Delta E_{\rm G}$ in its gravitational binding energy is,

$$\Delta E_{\rm G} = -{\rm G}\,M\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \tag{3}$$

where G is the universal gravitational constant. Evidently, as a result of contraction an amount of energy is released. This goes into increasing the random velocities of the constituent particles, raising the average temperature of the object[1]. The increased temperature increases the gas pressure and the system can find a configuration where the inward gravitational pressure is balanced by the outward gas pressure, i.e., a state of hydrostatic equilibrium.

[1]This is why gravitationally bound objects are known to have 'negative heat capacities'.

However, the increased temperature gives rise to enhanced radiative luminosity ($L \propto T^4$, L – surface luminosity, T – surface temperature) from the surface – a process through which the object lose the heat gained from gravitational contraction. Evidently, this reduces the temperature and the resultant gas pressure; eventually reaching a point when the gas pressure can no longer support the gravity. This results in an instability, inducing further contraction of the object and a repeat of the whole cycle itself [2].

[2]Kelvin and Helmholtz proposed this process (late 19th century) to explain the energy source of the Sun. However, the total energy that can be made available through this process ($E_{\rm G}^{\rm total} \sim {\rm GM}_\odot^{\ 2}/{\rm R}_\odot$) would last only about a few million years at the present rate of solar radiation, whereas fossil records indicate that the Sun has been shining at its present rate for a few billion years at least. It was shown much later (1930s) by Hans Bethe that the source of Sun's energy is nuclear, but that is a different story altogether.

1.2.2 ***Degeneracy Pressure***: As a self-gravitating object continues through phases of repeated contractions, its density keeps rising. When the density is such that the interparticle spacing ($\propto n^{-1/3}$, n – number density of particles) becomes comparable to the thermal de Broglie wavelength given by,

$$\lambda_{\rm deBroglie} \simeq \frac{h}{\sqrt{2\pi m k_B T}}, \tag{4}$$

(where m is mass of the particles, $\rm k_B$ is the Boltzmann constant and h is the Planck's constant), then quantum effects start having

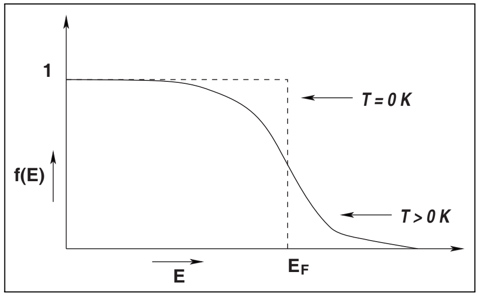

Figure 3. Fermion energy distribution near absolute zero of temperature.

an increasingly important role in the behaviour of the fluid. It is then that a new agent for resisting the gravity is found in the form of quantum effects [3].

[3]Onset of quantum effect : $n^{-1/3} \simeq \frac{h}{\sqrt{2\pi m k_B T}}$

When quantum effects become important, Fermions[4] follow a special energy distribution given by,

$$f(E) = \frac{1}{e^{\frac{E-E_{\rm F}}{k_{\rm B}T}} + 1}, \tag{5}$$

[4]Particles with half-integral spin following Fermi–Dirac statistics.

where $E_{\rm F}$ is known as the 'Fermi energy'. In particular, at absolute zero of temperature, Fermions fill up all the available energy states below $E_{\rm F}$ with one (and only one) particle in each, whereas the energy levels above $E_{\rm F}$ have zero occupation in them (*Figure* 3). Such a zero temperature Fermionic system is commonly known as a 'degenerate system'.

The Fermi temperature, defined as $T_{\rm F} = E_{\rm F}/{\rm k_B}$, provides an easy marker for the behaviour of a Fermionic system. When the physical temperature T of a system is much smaller than the Fermi temperature ($T << T_{\rm F}$), the behaviour of the system can be approximated to that of a system at (or near) absolute zero of temperature. Interestingly, quite a few astrophysical objects – Jovian planets, brown dwarfs, white dwarfs, and neutron stars, fulfill this criterion, and therefore can be treated as zero temperature Fermion systems.

For a non-relativistic, zero temperature Fermion system, the density of particles (with a specific momentum p) is given by,

$$n(p)\,dp = \frac{2.4\pi}{\mathrm{h}^3}\,p^2\,dp, \quad p \le p_{\mathrm{F}}, \tag{6}$$
$$n(p)\,dp = 0, \quad p > p_{\mathrm{F}}, \tag{7}$$

where p_{F} is the momentum corresponding to E_{F}. The pressure exerted by such a system is then,

$$P = \frac{m}{3}\int_0^{\infty} n(v)\,v^2\,dv = \int_0^{p_{\mathrm{F}}} \frac{p^2}{3m} n(p)\,dp = \left(\frac{3}{8\pi}\right)^{\frac{2}{3}} \frac{h^2}{5m} N^{\frac{5}{3}}, \tag{8}$$

where v $(= p/m)$ is the velocity, and N is the total number density of particles given by,

$$N = \int_0^{p_{\mathrm{F}}} n(p)\,dp = \frac{8\pi}{3h^3}\,p_{\mathrm{F}}^3\,. \tag{9}$$

When the physical temperature of a system is much smaller than the Fermi temperature, the behaviour of the system can be approximated to that of a system at (or near) absolute zero of temperature.

This rather important result for non-relativistic degenerate Fermi systems has a special significance for self-gravitating objects. Consider a self-gravitating object of mass M and radius R composed of a degenerate Fermi gas. From simple dimensional estimates we can find that the central gravitational pressure of such an object would be,

$$P_{\mathrm{G}} \simeq \frac{\mathrm{G}M^2}{R^4}\,. \tag{10}$$

If this is to be balanced by the degeneracy pressure of the Fermi gas then it implies that,

$$\frac{M^2}{R^4} \propto N^{5/3} \Rightarrow M\,R^3 = \mathrm{constant}\,, \tag{11}$$

(because N – the number density, scales as M/R^3) giving us one of the very important relations of planetary and stellar astrophysics. (Readers are advised to look up standard Statistical Mechanics texts [2, 3] for details of Fermion systems.)

2. Jovian Planets

Jupiter (*Figure* 4), named after the king of ancient Roman gods, is the largest planet of our solar system, and is thought to consist of

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a dense (rocky/icy) core with a mixture of elements, a surrounding layer of liquid metallic hydrogen with some helium, and an outer layer predominantly of molecular hydrogen. Evidently, the metallic hydrogen is largest component of this planet, and therefore plays the most significant role in the structure and stability of Jupiter. Saturn, the next largest planet, is also expected to have similar interior composition as Jupiter.

Both Jupiter and Saturn indicate the presence of a positive temperature gradient from the surface to the interior. Conceivably, the higher interior temperatures allow for a transition of the molecular hydrogen (found in the outer layers) to a metallic phase. This metallic state is expected to be quite similar to regular terrestrial metals in which the electrons form a degenerate Fermi gas. It is expected that at the end of their Kelvin–Helmholtz phase of contraction, both Jupiter and Saturn would settle down into a hydrogen-degenerate phase where the resistance to gravitational pressure would be supplied by the degeneracy pressure of the

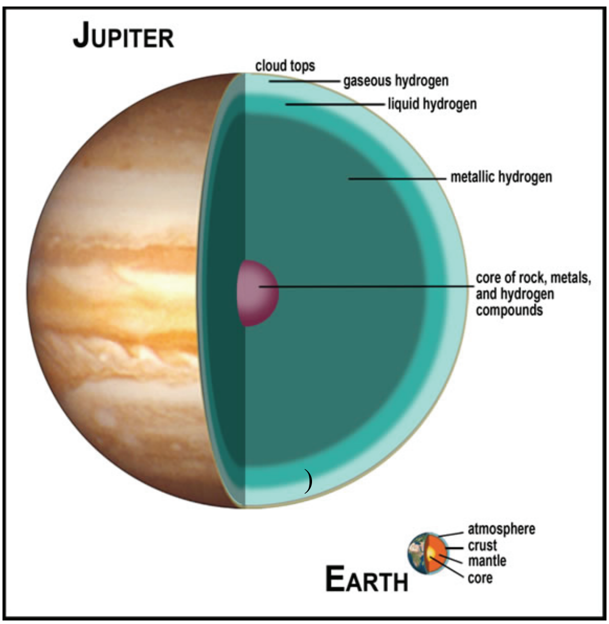

Figure 4. Interior of Jupiter as indicated by numerical simulations based on various observational data. (Image Source: `http://www.lpi.usra.edu/`.

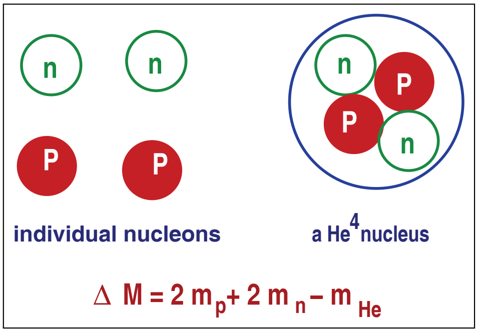

Figure 5. Energy generation in the nuclear reaction as a result of mass defect.

electrons of the metallic hydrogen[5].

However, observations indicate that both Jupiter and Saturn radiate away more energy than what they receive from the Sun. This clearly suggests that both of these planets are still in their Kelvin–Helmholtz phase of contraction and heat generation. Interestingly, the Kelvin–Helmholtz process appears not to account for the entire amount of radiation from Saturn, suggesting that it may have another source of power besides simple gravitational contraction.

[5]It needs to be noted that the pressure would also have a small but non-zero contribution from the hydrogen nuclei.

3. Brown Dwarfs

A star gets its energy from nuclear burning – combining four hydrogen (H) atoms into one helium (He^4) atom, for most of its life. The mass of He^4 happens to be slightly smaller than the mass of the four H atoms, giving rise to a small 'mass defect' (meaning, a deficit). It is this defect that shows up as energy through the famous mass-energy equivalence relation of Einstein ($E = \Delta M\, c^2$) (*Figure* 5).

Now, the conversion of H into He^4 actually happens in a number of steps, each requiring higher and higher ambient temperature for it to progress (*Figure* 6). When a self-gravitating object undergoes repeated phases of Kelvin–Helmholtz contraction, the final temperature reached depends on its mass. The higher the mass, the higher is the final temperature. Though quite heavy by planetary standards, the Jovian planets fall far short of the magic temperature (about a million degree Kelvin) to begin hydrogen

Interestingly, the Kelvin–Helmholtz process appears not to account for the entire amount of radiation from Saturn, suggesting that it may have another source of power besides simple gravitational contraction.

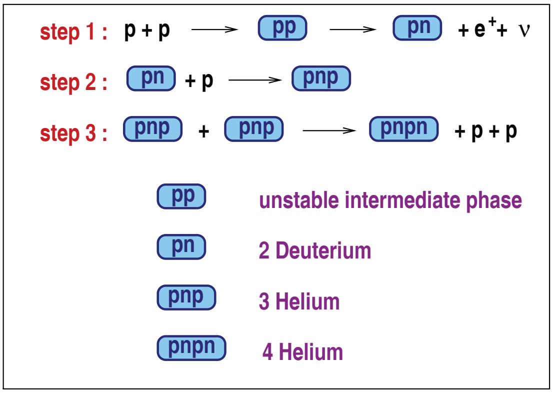

Figure 6. The $p-p$ chain, showing steps of conversion of hydrogen into helium. This is the process of energy generation active inside our Sun.

Brown dwarfs never reach the hydrogen fusion temperatures and therefore mostly retain their primordial lithium – a marker to distinguish candidate brown dwarfs.

fusion. Between the Jupiters and the stars, there exist another class of objects – the 'brown dwarfs', which are really 'failed stars'. They do achieve nuclear fusion, but not of hydrogen. Instead, brown dwarfs are known to burn deuterium (D^2, a heavier isotope of hydrogen).

In deuterium fusion, a deuterium nucleus and a proton combine to form a helium-3 (He^3) nucleus. This occurs at the second stage of the $p-p$ chain of the hydrogen fusion, but can also proceed from primordial deuterium. This onset of deuterium fusion separates the class of brown dwarfs from that of the Jovian planets, giving the maximum mass of gas giants to be about 0.012 $M_\odot$ (or about 13 Jupiter masses). Whereas, the class of brown dwarfs is separated from the class of stars by the threshold for hydrogen fusion at 0.08 $M_\odot$. Once the deuterium burning gets over, a brown dwarf continues to cool, and ultimately the pressure support comes from the electron degeneracy pressure of H and He^3.

A particularly important characteristic of brown dwarfs is the presence of primordial lithium in them. Stars, achieving hydrogen fusion temperature rapidly deplete their lithium. On the other hand, brown dwarfs (even massive brown dwarfs) never reach this temperature and therefore mostly retain their primordial lithium. The presence of lithium is therefore used to distinguish candidate brown dwarfs.

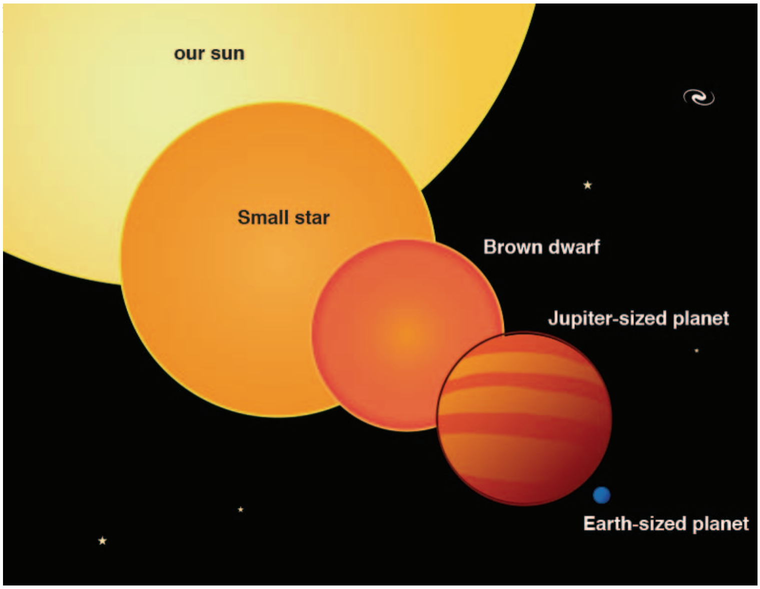

Figure 7. A schematic comparison of various stellar and sub-stellar objects. (Image Source: `https://spaceplace.nasa.gov/`

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My thanks to G Srinivasan for setting me on the path of Fermi-degenerate objects by explaining that the study of stellar evolution actually begins with Jupiter (among other such wonderful concepts that we'd encounter in the next installments of this series), and to D Narasimha for introducing me to the physics of stars one exciting summer, thirty years ago. I would also like to express my deep gratitude to the *Resonance* team for their patience and perseverance in getting this article published after I miserably failed all possible deadlines.

Suggested Reading

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