Think It Over

This section of Resonance presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to Think It Over, Resonance, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

Chat Time Sam!

Question 1.
(a) Show that there is no equilateral triangle on the plane whose vertices have integer co-ordinates.
(b) Show that there exist infinitely many equilateral triangles in 3D-space whose vertices have integer co-ordinates.
(c) Characterize all equilateral triangles in 3D-space whose vertices have integer co-ordinates.

Question 2.
(a) As $n$ varies over natural numbers, consider the sequence

$$a_n = \left[n + \sqrt{n} + \frac{1}{2}\right],$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Characterize, with proof, the numbers which appear in the sequence $\{a_n\}$.

(b) As $n$ varies over natural numbers, consider the sequence

$$b_n = \left[n + \sqrt{2n} + \frac{1}{2}\right].$$

Characterize, with proof, the numbers which appear in the sequence $\{b_n\}$.

(c) Let $r > 1$ be a positive integer. Characterize the sequence

$$c_n := \left[n + (n + n^{1/r})^{1/r}\right]$$
similarly.

Keywords
Equilateral lattice triangles, greatest integer function, recurrences, tiling by similar rectangles.
**Question 3.**
Let \( a_0 = 1, a_1 = 1, a_2 = 1/2 \) and \( n a_n = a_{n-1} + a_{n-3} \) for \( n > 2 \).
Define \( s_n \) as
\[
a_0^2 s_n + a_1^2 s_{n-1} + (2!)a_2^2 s_{n-2} + \cdots + (n-1)!a_{n-1}^2 s_1 = n(n!)a_n^2.
\]
The first few terms of \( \{s_n\}_{n \geq 1} \) are 1, 0, 4, 8, 5, 36, 98, 112, 490, \ldots.
Determine all \( n \) such that \( s_n \) is odd.
(the number \( s_n \) is the number of subgroups of index \( n \) in the so-called free product of the group of order 3 with itself).

**Question 4.**
Prove that every closed continuous curve on the plane of unit length can be enclosed by a rectangle of area \( 1/4 \).

**Question 5.**
Let \( t \) be a positive, real number. We wish to tile (that is fill without overlapping) a square with rectangles which are similar to a \( 1 \times t \) rectangle.
Show (with proof) that such a tiling is:
(a) possible when \( t = 3 \times \sqrt{3} \) and
(b) not possible when \( t = \frac{3}{\sqrt{2}} \).