Golden Ratio

A Measure of Physical Beauty

Syed Abbas

Our attraction to another body increases if the body is symmetrical and in proportion. If a face or a structure is in proportion, we are more likely to notice it and find it beautiful. The universal ratio of beauty is the 'Golden Ratio', found in many structures. This ratio comes from Fibonacci numbers. In this article, we explore this concept along with its applications.

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Beauty is a relative concept and it varies from region to region and person to person. As mathematics gives precise definition of everything, so it does for beauty. Leonardo da Vinci's drawings of the human body emphasized its proportion. The ratio of the following distances is the 'Golden Ratio' – (foot to navel) : (navel to head). Similarly, buildings are more attractive if the proportions follow the golden ratio. The golden ratio (or 'golden section') is based on Fibonacci numbers[1], where every number in the sequence (after the second) is the sum of the previous two numbers. Let us consider the example of a mask of the human face based on the golden ratio. We keep the proportions of the length of the nose, the position of the eyes and the length of the chin, all to some aspect of the golden ratio [1]. When we place this mask over the photo of a human face it would be seen that there is a good fit. Good fit means that the proportions of the face fits in geometrically 'nice' proportions with the mask, based on the golden ratio. In this case, the face is accepted as mathematically beautiful. Stephen Marquardt has studied human beauty for years in his practice of oral and maxillofacial surgery, and he developed a mask using the concept of golden ratio. The mask is called the Marquardt beauty mask (*Figure* 1) [1].

[1] See Somnath Basu, Fibonacci numbers, seating arrangements and stair-climbing, *Resonance*, Vol.7, No.1, pp.78–87, 2002.

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There are several examples of golden ratio throughout the designs in nature. Let us take a look at one of the most important namely,

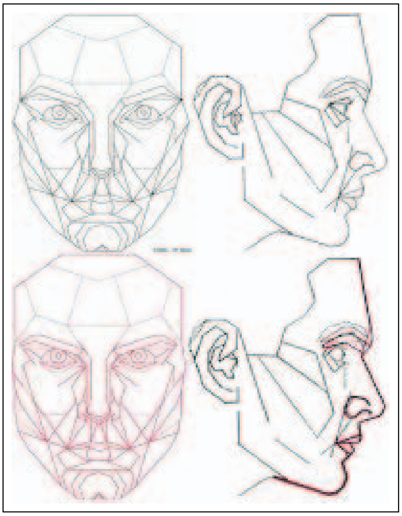

Figure 1. Golden ratio mask [1].

the human body. The human body illustrates the golden ratio or the divine proportion. According to the book by Samuel Obara [2], the following proportions in the human body are very close to the golden ratio (*Figure* 2):

- Height of a person, and distance between the naval point and the foot.
- Length between the pupils, and length between the eyebrows.
- Width of the nose, and length between the nostrils.

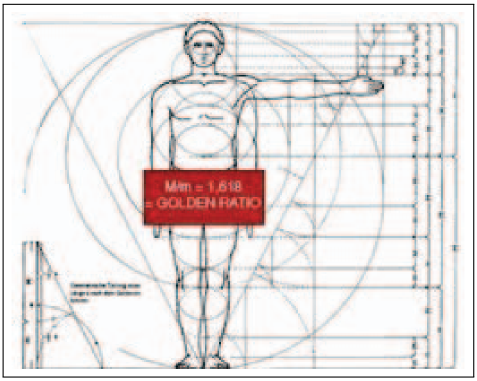

Figure 2. Proportions in the human body [2].

- Length between the shoulder line, and length of the head.
- Distance from the ground to the naval point divided by distance from the ground to the knee.
- Distance from the wrist to the elbow divided by length of the hand.

Golden ratio comes from Fibonacci numbers, so now we briefly describe these numbers. The Fibonacci numbers F_n in mathematical terms are defined by the following recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the beginning values $F_1 = 1$, $F_2 = 1$ or $F_0 = 1$, $F_1 = 1$. Fibonacci sequence is named after the famous mathematician–Fibonacci. It has been observed that these numbers appeared in Indian mathematics with Sanskrit prosody (poetry) well before. Pingala was the first to write a treatise named *Chandahsastra* based on metres in Sanskrit prosody [3]. His work contains basic ideas of Fibonacci numbers. The modern theory in terms of explicit mathematical terms appeared in the book *Liber Abaci* written in 1202 by Leonardo of Pisa, also known as Fibonacci. Fibonacci introduced the sequence to the Western European mathematics, where the sequence begins with $F_1 = 1$, without an initial 0. Fibonacci considered the growth of an idealized but not very biologically realistic rabbit population. He assumed a newly born pair of rabbits, one male and one female, in a field. Since the rabbits are able to mate at the age of one month, by the end of the second month, the female can produce another pair of rabbits. For the sake of this model, he assumed that the rabbits never die. Thus, a mating pair always produce one new opposite sex pair every month from the second month onwards. If we start counting the number of rabbit pairs, we get Fibonacci numbers.

Now we will explain how Fibonacci numbers occur in Sanskrit prosody. In the Sanskrit tradition of prosody, there was an interest in enumerating all patterns of long **L** syllables that are 2 units of duration, and short **S** syllables that are 1 unit of duration. Counting the different patterns of **L** and **S** of a given duration results in Fibonacci numbers. Actually the number of patterns that are m short syllables long is the Fibonacci number F_{m+1}. Let us explore

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this fact more. There is a nice article in *Historia Mathematica* by Parmanand Singh [3] in which the author has explained in detail the occurrence of Fibonacci numbers in ancient and medieval Indian texts. We know that there are stanzas with four quarters in Sanskrit prosody. In each quarter, there may be same number of syllables, or the same number of time units, a long one and a short one. Moreover in Sanskrit prosody, the metres are generally categorized into three. The varna-vrttas are metres in which the number of letters are constant and the number of units of duration are arbitrary. The matra-vrttas are metres in which the number of units of duration are constant and the number of letters are arbitrary. The last category is the metres called gana-vrttas, which have groups of units of durations. In this, the number of units of durations in the group is constant, and the number of letters in the group is arbitrary. The metre of three units of durations have number of variation three. One can obtain the expansion by writing the variation of the metre of one units of duration on the left of the **L** and then each variation of two units of duration on the left of the **S**. For metres having more than three units of duration, one can follow the same procedure. The procedure is depicted in *Figure* 3. It is evident that the expansion of a metre of n units of durations can be obtained by writing each variation in the expansion of a metre of $n - 2$ units of duration, on the left of the **L**, and then same $n - 1$ units of durations on the left of the **S**. Hence the variation of matra-vrttas forms the sequence of numbers which are actually the famous Fibonacci numbers. The number of variations of metres having $1, 2, 3, 4, 5, 6, ..,$ units of durations are $1, 2, 3, 5, 8, 13, ...,$ which are nothing but Fibonacci numbers. We can also observe that it not only gives Fibonacci numbers but also the recurrence formula – $F_{m+1} = F_m + F_{m-1}$. This fact is illustrated in *Figure* 3.

From the above recurrence relation, the numbers $\{1, 1, 2, 3, 5, 8, 13, 21, ...\}$ forms a Fibonacci sequence. Here **L** is assigned for two units of duration and **S** is assigned for one unit of duration. So patterns with one short mora (syllabus) is only one **S** and patterns with two short morae are **L** and **SS**, and in terms of

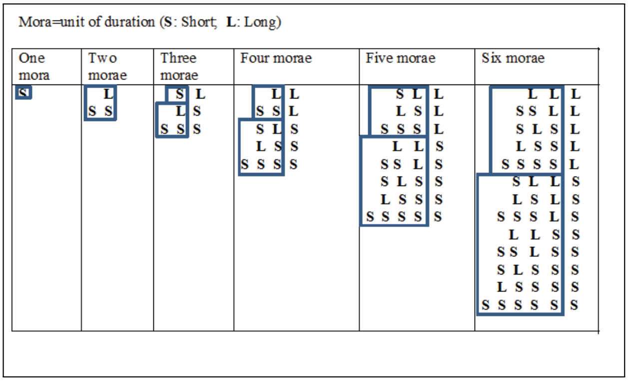

Figure 3. Fibonacci numbers in Sanskrit prosody [3].

numbers are 2 and 1 + 1. So if we represent these in numbers then their sum should be equal to number of morae. For example for there morae, the total sum of numbers assigned to letters **L** and **S** should be three. So the only possible combinations are $2 + 1, 1 + 2, 1 + 1 + 1$, which can be represented in terms of letters as **SL** (21), **LS** (12), **SSS** (111). We can similarly find patterns for four, five and six morae, which are illustrated in *Figure* 3.

When each number in the Fibonacci sequence is divided by its predecessor, the number converges to the golden ratio. For example, $\frac{5}{3} = 1.67$, $\frac{21}{13} = 1.61$, etc. Then the ratio between the obtained numbers gradually become close to 1.618. The golden ratio is actually the number $\frac{1+\sqrt{5}}{2}$ which is close to 1.618. The Binet's formula gives a closed form solution of sequence of Fibonacci numbers. The formula was already given by Abraham de Moivre, and is expressed as follows:

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}, \text{ where } \phi = \frac{1 + \sqrt{5}}{2} \approx 1.618033988\cdots,$$

and,

$$\psi = \frac{1 - \sqrt{5}}{2} = 1 - \phi = -\frac{1}{\phi} \approx -0.618033988\cdots,$$

where ϕ is the golden ratio. From the above expression, using $\psi = -\frac{1}{\phi}$, the formula can also be written as $F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$. We can establish it, since both ϕ and ψ are solutions of,

$$x^2 = x + 1, \quad x^n = x^{n-1} + x^{n-2}.$$

Hence,

$$\phi^n = \phi^{n-1} + \phi^{n-2}, \quad \psi^n = \psi^{n-1} + \psi^{n-2}.$$

Thus the sequence $\{\Phi_n\}$ defined by $\Phi_n = a_1\phi^n + a_2\psi^n$ satisfies the following recurrence relation,

$$\Phi_n = a_1\phi^n + a_2\psi^n = a_1\phi^{n-1} + a_1\phi^{n-2} + a_2\psi^{n-1} + a_2\psi^{n-2} = \Phi_{n-1} + \Phi_{n-2},$$

for any values of a_1 and a_2. Since the above expression is true for any values of a_1 and a_2, we can always choose them such that $\Phi_0 = 0$ and $\Phi_1 = 1$. In this case, the obtained sequence is actually the Fibonacci sequence. Mathematically, this can be stated as solving equations $a_1 + a_2 = 0$, $a_1\phi + a_2\psi = 1$ for a_1 and a_2, which we know is solvable, and the solution is given by $a_1 = \frac{1}{\phi-\psi} = \frac{1}{\sqrt{5}}$ and $a_2 = -a_1$.

Let us discuss few more problems where golden ratio occurs. Take a rectangle of sides in the ratio $1 : a$. Partition this rectangle into a square and a new rectangle such that the new rectangle has sides in the ratio $1 : a$. This problem gives a unique solution for a, which is interestingly nothing but the golden ratio. A rectangle having such a property is called the 'Golden Rectangle'[2]. Writing the above mentioned problem mathematically, we obtain $\frac{a}{1} = \frac{1}{a-1}$, which gives $a^2 - a - 1 = 0$. After solving this quadratic equation, we get $a = \phi = \frac{1+\sqrt{5}}{2}$.

[2]See Utpal Mukhopadhyay, Logarithmetic spiral – A splendid curve, *Resonance*, Vol.9, No.11, pp. 39–45, 2004.

We can also represent the golden ratio in terms of nested radicals. For example, if we define $a_n^2 = a_{n-1} + 1$ with $a_1 = 1$, the limit of sequence $\{a_n\}$ is the golden ratio. If we assume that a is the limit, then $a_n, a_{n-1} \to a$ as $n \to \infty$, which gives $a^2 = a + 1$, whose solution is the golden ratio. In fact, it can be expressed as $\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$. Moreover, the continued fraction representation is $1 + \frac{1}{1+\frac{1}{1+\cdots}}$. So it is the worst real number for rational approximation as it takes more running time.

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The Fibonacci numbers have great importance in several fields. For example, it is important in the computational runtime analysis of Euclid's algorithm to determine the greatest common divisor of two integers. In this case, the worst case input are pairs

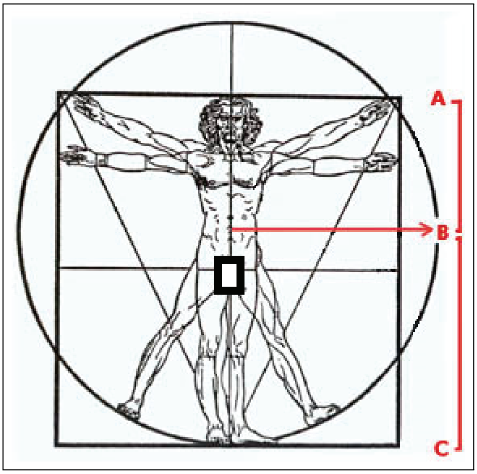

Figure 4. A painting by Da Vinci that follows the golden ratio [2].

of consecutive Fibonacci numbers. An important property of Fibonacci numbers is that any positive integer can be written as a sum of Fibonacci numbers, where any one number is used once at most. For example, $6 = 1 + 2 + 3,\ 7 = 2 + 5,\ 9 = 1 + 3 + 5$ or $1 + 8$, and so on.

There are many interesting facts related to the golden ratio. We mention few of them here.

Interesting Facts

There are several proofs that the golden ratio has been used by architects and artists throughout history to produce objects of great beauty. An example is the painting called the *Vitruvian Man* by Da Vinci (*Figure* 4). It has been observed that the Greek temples have the concept of golden ratio inherit in them. Golden ratio occurs in nature as well. The main reason behind this is that the best way to efficiently pack things tightly together is using the Fibonacci sequence.

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Fibonacci sequences appears in many biological settings. For instance, branches in trees, the arrangement of leaves on a stem, the arrangement of a pine cone, the fruitlets of a pineapple, the family tree of honeybees, etc., (*Figure* 5). Now we shall see how to draw the Fibonacci curve[3] depicted in these figures. We first need to draw a square of 1×1 unit. Then draw another square with

[3] See A K Mallik, From natural numbers to numbers and curves in nature – I & II, *Resonance*, Vol.9, Nos.9 and 10, 2004.

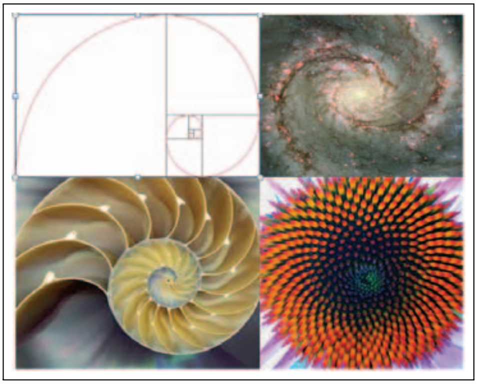

Figure 5. Fibonacci in nature [2].

same dimensions to right of this square. Next draw a 2×2 square above it, and then a 3×3 square on the left side of this 2×3 rectangle. After this, at each step, following counterclockwise, draw a square of the order of the height and then the next square of the order of the base of the previous rectangles. Proceeding in this manner, we get squares of order $1, 1, 2, 3, 5, 8, 13$, and so on. Now if we draw a curve starting from the leftmost corner of the first 1×1 square, passing through the corners of other squares in a counterclockwise pattern, then we get the curve depicted in *Figure* 5.

Also a recent interesting study by Aiden Dwyer claims that if we arrange a solar panel in the form of a tree where the ratio[4] follows the Fibonacci pattern, then the efficiency is improved very much [4,5]. His setup is shown in *Figure* 6 [4]. It has been observed that this tree design panel made 20 percent more electricity compared to a flat panel. Moreover, it collected two and a half hours more sunlight during the daytime. More interestingly, during winters it performed better compared to other flat panels and made 50 percent more electricity. Hence, it is clear that the concept of golden ratio can be very useful in improving the efficiency of several technologies.

[4] Start with a branch, which after some time period splits into two smaller branches – a main one and a sapling. In the next time period, the sapling stays the same size as it grows to adulthood, while the main branch once again splits into two. If we track this number, we get Fibonacci sequence. Also the ratio of length of the full branch and the length upto the point of splitting is the golden ratio.

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Golden ratio is eye catchy. Hence it is observed that the logos of famous brands, for example, Pepsi, Toyota, National Geographic, etc., follows the rule of golden ratio (*Figure* 7) [6].

Figure 6. Setup by Aiden Dwyer [4].

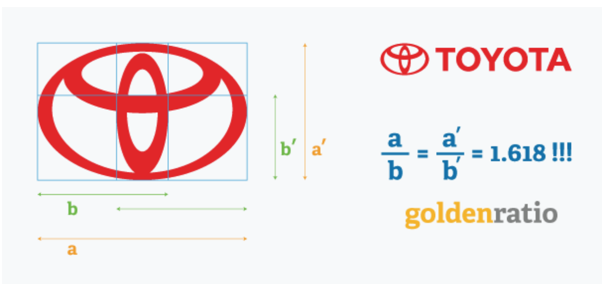

Figure 7. Toyota logo follows the golden ratio [5].

Open Problems

There are still some open questions to answer related to Fibonacci numbers, which are as follows.

A prime number which occurs in a Fibonacci sequence is called a Fibonacci prime. For example, the first few such primes are:

$$2, 3, 5, 13, 89, 233, 1597, 28657, 514229, \cdots .$$

Till date Fibonacci primes with thousands of digits have been found. But it is still not known whether there are infinitely many.

So it is still an open question for the scientific community. Also if the members of the Fibonacci sequence are taken over $mod(n)$, then the resulting sequence must be periodic with period at most $n^2 - 1$. The Pisano periods are defined by the sequence of the lengths of the periods for various n. Determining the Pisano periods in general is still an open problem. It can be only solved for any particular n using the instance of cycle detection.

Acknowledgement

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Suggested Reading

[1] www.goldennumber.net/beauty/3.
[2] Samuel Obara, *The Golden Ratio in Art and Architecture*, J. Wilson, University of Georgia, Department of mathematical education.
[3] Parmanand Singh, The So-called Fibonacci numbers in Ancient and Medieval India, *Historia Mathematica*, Vol.12, pp.229–244, 1985.
[4] www.amnh.org/learn-teach/young-naturalist-awards/winning-essays2/2011-winning-essays/thesecret-of-the-fibonacci-sequence-in-trees
[5] www.redicecreations.com/article.php?id=16548
[6] http://thecuriouscan.com/golden-ratio-the-mathematics-behind-awesome-logos-that-youshould-know
[7] www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html

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