Tables Turned: Physics in the Service of Mathematics

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E T Bell, the famous author of '*Men of Mathematics*', has described mathematics as the 'Queen of Arts and Servant of Science'. What he meant is that mathematics serves science by entering into the picture as soon as a proper mathematical model is set up by the scientist, and then after a purely mathematical analysis of the model, the final mathematical step is interpreted scientifically. The purpose of the present article is to convince the readers that sometimes the roles of science and mathematics are reversed, and a mathematical problem is interpreted as a physics problem; the laws of physics are utilized for a physical analysis, and the final result of the physical analysis is interpreted mathematically. We shall illustrate this by means of few examples.

Problem 1: *Given three non-collinear points A, B, and C, find the point X such that X is coplanar with the plane determined by A, B, C and $XA + XB + XC$ is least.*

Physical Approach: Imagine a tall table having holes at the points A, B and C. Take three strings of length exactly the height of the table[1] . Pass each string through one of the holes, and tie the top ends of the strings together. The other ends of the strings hang in the air directly below A, B and C. Hang equal weights under the table to the ends of the strings. The arrangement looks as in *Figure* 1.

[1]The height of the table does not play a significant role in the problem.

The potential energy of the system of weights is $P = 1 \cdot a \cdot g + 1 \cdot b \cdot g + 1 \cdot c \cdot g$ where a, b, c are the heights of the weights hanging at A, B and C respectively. But $a = AX$, $b = BX$, $c = CX$. Hence $P = g(AX + BX + CX)$. For the system to be in equilibrium, P must be minimal, i.e., $AX + BX + CX$ should be minimal. Hence X, the position of the common top end of the three strings is the point we are seeking. But we don't want to drill holes on the tops of real tables to find X!

Mathematics has been described as the 'Queen of Arts and Servant of Science'.

Keywords
Collinear, non-collinear, congruent, electrical network, tiling, Fermat point.

So, let us devote some more of our time to this problem! We

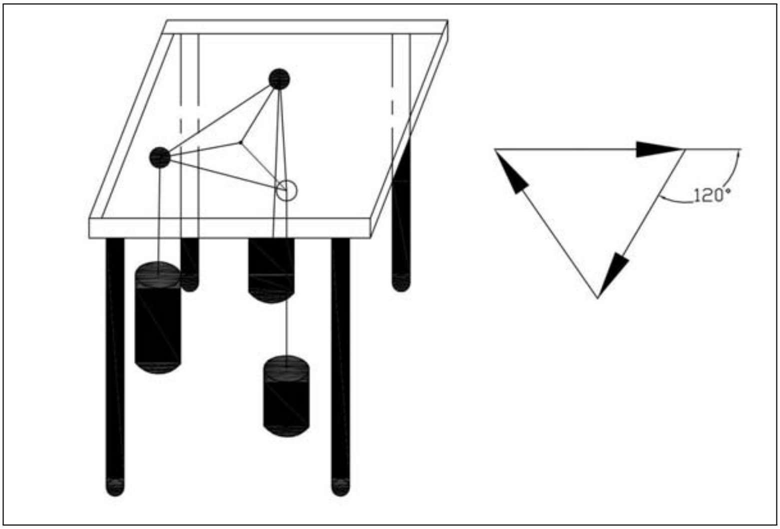


Figure 1. Fermat point of a triangle.

know that the point X is in equilibrium and hence the tensions in the strings XA, XB and XC should add up to zero. If three forces add up to zero then they should form a triangle PQR such that $\overrightarrow{PQ}, \overrightarrow{QR}$ and $\overrightarrow{RP}$ represent the forces. This justifies the drawing of the triangle on the right of our table in *Figure* 1. Since the weights are all equal, the tensions are all equal in magnitude and hence our triangle is equilateral.

Hence,

$$\angle AXB = \angle BXC = \angle CXA = 120^{\circ} \tag{1}$$

The role of Physics ends here. Having found out that X is a point that is such that $\angle AXB = \angle BXC = \angle CXA = 120^{\circ}$, the question arises as to how to find such a point. For this, we employ Euclidean geometry.

On each of the sides AB, BC, and CA of ABC, erect equilateral triangles ABC', BCA', CAB' out of ABC (*Figure* 2).

Join AA', BB' and CC'. Let BB', CC' intersect at Y.
Look at triangles ABB' and $AC'C$:
$AB = AC', AB' = AC, \angle BAB' = \angle A = \frac{\pi}{3} = \angle C'AC$.

By SAS property, the triangles are congruent.

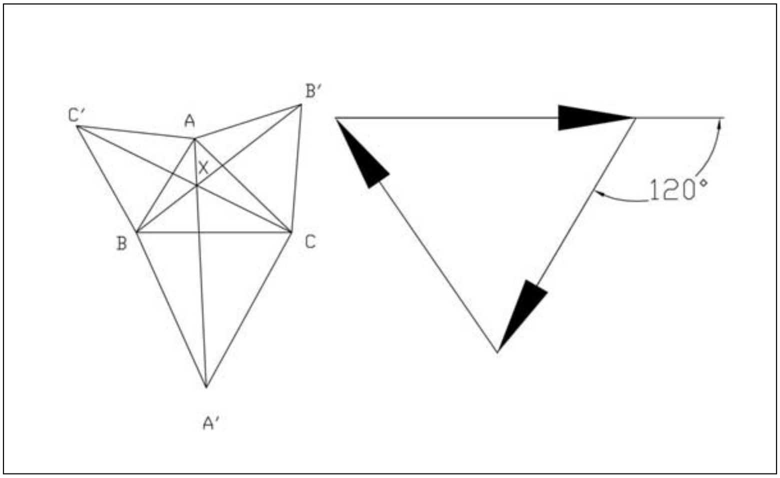


Figure 2. A geometric construction for the Fermat point of a triangle.

Hence $\angle AC'C = \angle ABB'$.

Hence $\angle AC'Y = \angle ABY$. Since AY subtends equal angles at B and C', it follows that A, B, C', X are concyclic. Since $\angle AC'B = 60^{\circ}$, it follows that $\angle AYB = 120^{\circ}$ Similarly $\angle AYC = 120^{\circ}$. In quadrilateral $BYCA', \angle BYC = 120^{\circ}$ and $\angle BA'C = 60^{\circ}$. Hence B, Y, A', C are concyclic. So CA' submits equal angles at X, B. Hence $\angle CYA' = \angle CBA'$. But $\angle CBA' = 60^{\circ}$ as BCA' is equilateral. So $\angle CYA' = 60^{\circ}$.

Therefore $\angle AYC + \angle A'YC = 20^{\circ} + 60^{\circ} = 180^{\circ}$. This shows that A, Y, A' are collinear. Therefore AA', BB', CC' are concurrent at X.

Now,

$$\angle BYC = 360^{\circ} - \angle AYC - \angle AYB = 360^{\circ} - 240^{\circ} = 120^{\circ}. \quad (2)$$

So Physics leads us to a point X such that $\angle AXB = \angle BXC = \angle CXA = 120^{\circ}$, and mathematics leads us to a point Y such that $\angle AYB = \angle BYC = \angle CYA = 120^{\circ}$. Are we sure that $X = Y$? It will be hasty to be sure! We need to prove it.

Suppose X and Y were different. Then the picture looks like this (*Figure* 3).

$\angle XAB + \angle XBA < \angle YAB + \angle YBA = 60^{\circ} \Rightarrow \angle AXB > 120^{\circ}$, a

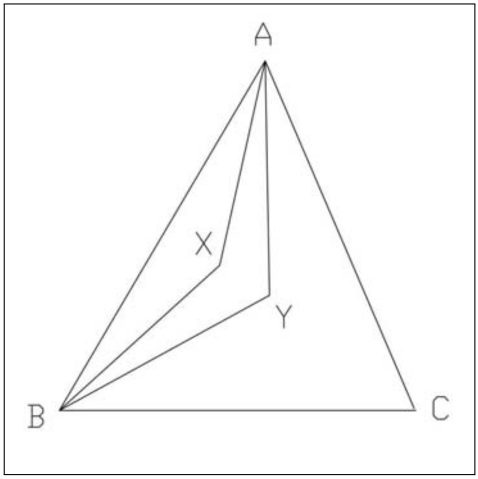


Figure 3. Uniqueness of the Fermat point.

contradiction.

Therefore $X = Y$.

So finally we have proved that X can be obtained by constructing equilateral triangles on sides outside ABC and joining AA', BB', CC' to obtain their point of concurrence.

The point is variably known as Fermat Point, Toricelli Point and Fermat–Toricelli Point. We prefer the last name because the problem was originally posed by Fermat and solved by Toricelli!

An Important Comment:

Look at (2): $\angle AXB = \angle BXC = \angle CXA = 120^{\mathrm{o}}$. So we have the following picture (*Figure* 4).

Discovery of a subtle mistake; figures could be deceptive.

Produce AX to intersect BC at D. Then $\angle BXD > \angle BAX$ by the exterior angle theorem. $\angle CXD > \angle CAX$ for a similar reason. Adding these inequalities, we obtain $120^{\mathrm{o}} = \angle BXC > \angle BAC$. In a similar way we can prove that $\angle B < 120^{\mathrm{o}}$ and $\angle C < 120^{\mathrm{o}}$. But there is no reason why each of the three angles of a triangle should be less than 120^{o}. This shows that we commited a mistake somewhere though our arguments appear to be sound. We committed a mistake which is very subtle. The mistake lies in the figure that we started with! Our figure and psyche misled us into the belief that the Fermat point X of the triangle ABC lies inside

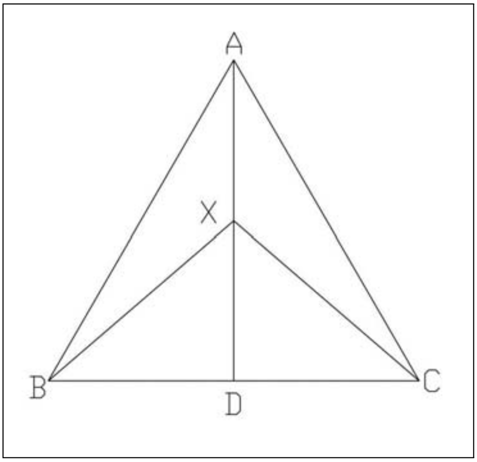


Figure 4. Figure for the variant of the Fermat point problem.

ABC. Nowhere have we proved that X cannot be situated on or outside ABC.

We leave it to the reader to show that X cannot lie on or outside the triangle. A variant of the above problem is the following:

Problem 2: *There are three schools located at three non-collinear points A, B and C having, on an average, a, b, c students respectively. Where should a school be located if it is to serve the children of all the three villages*?

Solution: Referring to our table model, we hang weights of a, b, c kilograms to the threads passing through A, B, C respectively. Then the potential energy is $P = G(aAX + bBX + cCX)$. If one of the angles of triangle ABC say $\angle A$ is equal to 120° then $X = A$. P must be minimal. Otherwise X, the position of the common top end of the strings is the point that minimizes $aAX + bBX + cCX$, and hence X is the point we are seeking. Also the tensions in the strings should add up to zero, and hence they should form a triangle whose sides measure a, b and c. As in the previous problem, the exterior angles of this triangle form the angles between the strings (*Figure* 5).

The exterior angle α=angle between AX and BX, and two more similar statements are also true. It is easy to check that differential calculus too leads to the same conclusion. Just imitate the steps

Figure 5. Angle α between AX and BX.

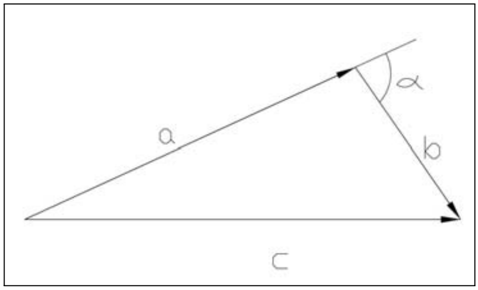
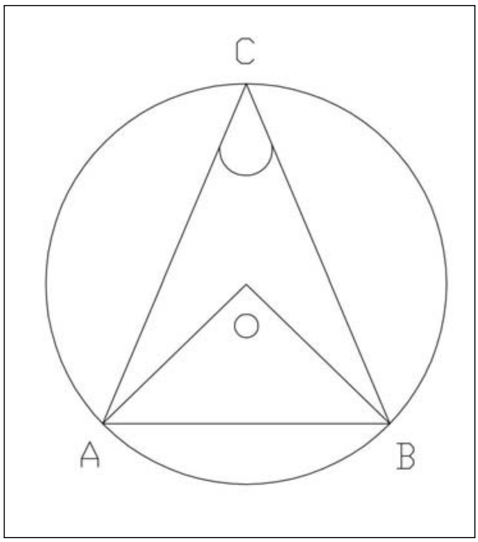


Figure 6. Locating X.



in the previous problem. So now, we have to locate the point X, from knowledge about angles AXB, BXC, and CXA.

Now all points X such that $\angle AXB = \alpha$, lie on a circular arc (*Figure* 6).

From pure geometry we know that AB subtends an angle of 2α at the center of the circle of which AB is a chord and $\angle AYB = \alpha$ for all Y on an arc with end points A, B. So $\angle AOB = 2\alpha$. Since $OA = OB =$ radius, $\angle OAB = \angle OBA = \frac{\pi}{2} - \alpha$. So we draw lines at A and B making an angle of $\frac{\pi}{2} - \alpha$ with AB and mark their intersection as O. O is the center of the circle we are seeking.

With O as center and OA as radius draw a circle. We know that X is a point on this circle. Follow the same procedure to draw a circle passing through B and C such that X is a point on this circle. This circle and the earlier circle have B and X as their common points. So we found X.

Tiling abhors irrationality.

Tiling: We consider the following question [1]: *Given a rectangle of sides m and n, under what conditions on m and n is it possible to tile the rectangle with congruent squares?*

Solution: Suppose the rectangle can be tiled with congruent squares of side a. Then $m = pa$ and $n = qa$ for some positive integers p, q and so $\frac{m}{n} = \frac{p}{q}$ is rational. Conversely, suppose $\frac{m}{n}$ is rational. Let $\frac{m}{n} = \frac{p}{q}$ where p, q are positive integers. Then $\frac{m}{n} = \frac{p}{q} = a$ (say). A square of side a can be used to tile the rectangle. The rectangle can be tiled using pq squares in such a way that each row of the squares has m of them and each column has n of them.

Remark: The above discussion shows that a rectangle for which $\frac{m}{n}$ is irrational cannot be tiled with congruent squares. At this juncture, we ask the reader to appreciate the fact that this is a 'theoretical impossibility' because, by the density of the set of rationals in the real number system, it is possible to select a rational number $\frac{p}{q}$ which is so close to $\frac{m}{n}$ that no existing (or even prospective) measuring instrument can detect the difference between $\frac{p}{q}$ and $\frac{m}{n}$, and we can tile the rectangle supposing $\frac{m}{n}$ to be the rational number $\frac{p}{q}$.

A problem solved leads to the creation of a tougher problem. The above remark raises the following question.

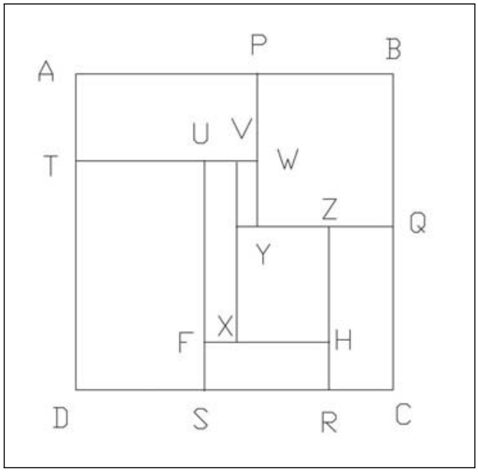
Problem 3: *Which rectangles can be tiled with squares if the restriction that all the squares involved in the tiling be congruent is dropped?*

An Electrifying Solution: Before attempting to solve the tiling problem, it is useful to recall Ohm's law and Kirchoff's current law from electricity.

Kirchoff's laws in the service of geometry.

Ohm's Law: *If a conductor has voltages V_1, V_2 at its ends with $V_1 > V_2$ and has conductance C (= reciprocal of resistance), then*

Figure 7. A direction.



the current flowing through it is $I = (V_1 - V_2)\,C$ and the current flows from the V_1–end to the V_2–end.

Kirchoff's Current Law: *In any electrical network, the total current entering a node is the same as the total current leaving it.*

Now imagine a rectangle tiled with (smaller) rectangles (*Figure* 7). For example, look at the horizontal line AB. From AB hang two rectangles $APWT$ and $PBQY$. The sum of the horizontal dimensions of $APWT$ and $PBQY$ is $AP + PB$. Now look at the bottom horizontal lines TW and YQ of the rectangles. The horizontal lines resting on them are TU, UV, VW, YZ, ZQ. The sum of these is $TU+UV+VW+YZ+ZQ$ which is the same as $AP+PB$. This observation paves the way for translating our dissected rectangle into an electrical network. Each horizontal line could be made into a node, and the horizontal dimensions could be made into currents in the conductors of the network to be formed. To make matters more clear, we explain through an example (see *Figures* 8, 9).

Steps to make the network.

Step 1: Label each horizontal line as $A, B, C, ...$, etc. These are the vertices.

Step 2: With each rectangle of the tiling we associate a conductor

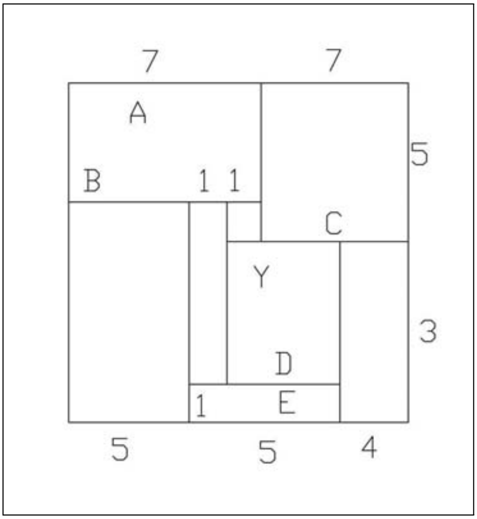


Figure 8. A concrete example of a dissection.

as follows. If the top and bottom sides of the rectangle lie on horizontal lines labelled V_1 and V_2 then V_1 and V_2 are joined by a conductor.

Step 3: The current in a wire shall be h if its associated rectangle has horizontal dimension h.

Step 4: The voltage at each node is the height of the node above the bottom most horizontal line.

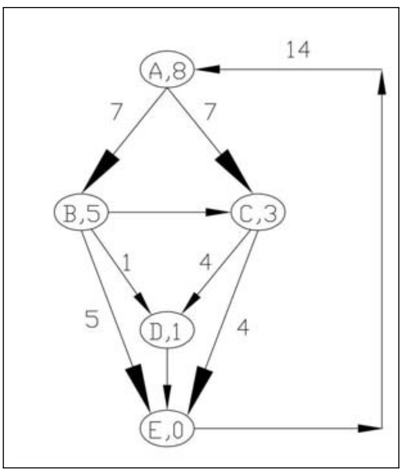


Figure 9. The network corresponding to the dissection in *Figure* 8.

The network (*Figure* 9) has been created for our example. Since current flows from a high voltage end to a low one, the directions of the current can be marked on the conductors. A number by the side of a node denotes the voltage of that node. Let us now go back to our question, *what sort of rectangles can be tiled with squares (not necessarily congruent)*? We shall now show that even when non-congruence of the tiles is allowed, an irrational ratio of the dimensions disallows tiling by squares. Suppose $\frac{m}{n}$ is irrational. Tiling by squares of an $m \times n$ rectangle is possible if tiling by squares of an $(\frac{m}{n}) \times 1$ rectangle is possible (just reduce or magnify by a factor of n to obtain a tiling of the other rectangle). Let $\frac{m}{n} = a$ be irrational. If an $a \times 1$ rectangle could be tiled by squares then conductance in each wire would be $\frac{\text{current}}{\text{voltage}} = \frac{\text{horizontal length}}{\text{vertical length}} = 1$. Now choose our conductors to have conductances equal to 1, and choose a voltage source such that input current is equal to 1. Since the input parameters are all numerically equal to 1, and since all calculations using Ohm's law and Kirchoff's Current law involves only the operations of $+, -, \times$ and $\div$, all the voltages must be rational. In particular, the voltage at the highest point cannot be irrational. This proves that an $m \times n$ rectangle with $\frac{m}{n}$ irrational cannot be tiled with squares.

The reader is strongly recommended to refer [1] and [2] for an excellent discussion of relevant topics.

The Pythagoras Theorem: Let a, b be the lengths of the sides and c the length of the hypotenuse of a right-angled triangle. Imagine a tub with the given right triangle as its base (*Figure* 10). We mount the tub so that it can freely rotate about a vertical axis passing through one end of the hypotenuse. Any push on a side of the tub will cause a rotation of the tub. Fill the tub with water. The water pushes each of the three faces outward. If these three pushes on the faces do not balance out then the tub would rotate on its own forever, violating the principle of conservation of energy. Hence the forces on the walls of the tub must balance out.

Pythagoras and the tub; not Archimedes and the tub!

Let the force of pressure per unit length of the wall be P kg wt/cm. Then the force on AC is cP, that on BC is bP, and that on AB is

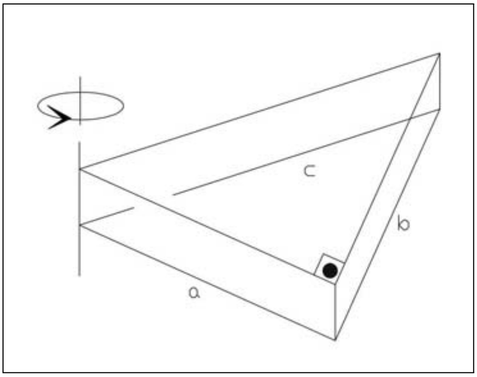


Figure 10. A physical setup for demonstrating Pythagoras theorem.

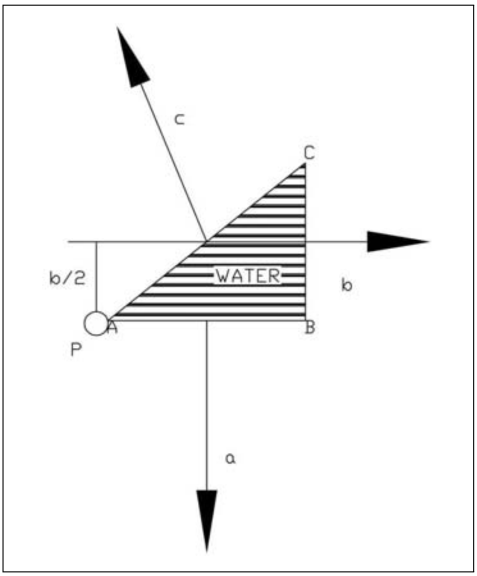


Figure 11. Force diagram for the setup in *Figure* 10.

aP (*Figure* 11). The points of application of the forces may be thought of as the mid points of AB, BC, CA respectively, since the center of gravity is the geometric center in the case of uniform masses. The moment of the force aP is $aP \times \frac{a}{2} = \frac{a^2P}{2}$. The moment of the force bP is $bP \times \frac{b}{2} = \frac{b^2P}{2}$. The moment of the force cP is $cP \times AD$ where D is foot of the perpendicular from A onto the line of action of cP. But $AD = \frac{c}{2}$. Hence moment of the force cP is $\frac{c^2P}{2}$. Since the net torque on the tub is zero, $\frac{c^2P}{2} - \frac{a^2P}{2} - \frac{b^2P}{2} = 0$ or $a^2 + b^2 = c^2$.

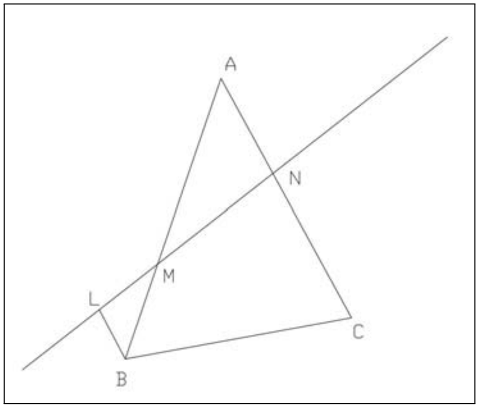


Figure 12. Moment of bC.

In fact, we can prove the more general cosine rule by starting with a general triangle. To see this, imagine the same type of arrangement as above except that ABC need not be right-angled (*Figure* 12).

The only difference that arises is in the value of the perpendicular distance from B to the line of action of the force on AC. Earlier, when $\angle C = 90^o$, BL was $\frac{b}{2}$. Note that $\angle BML = \angle AMN = 90^o - A$ and hence $\angle MBL = A$. So $\cos A = \frac{BL}{BM}$ or $BL = BM \cos A = (c - AM) \cos A = c \cos A - AM \cos A = c \cos A - AN = c \cos A - \frac{b}{2} = \frac{1}{2}[2c \cos A - b]$. As earlier, $\frac{c^2 P}{2} = \frac{a^2 P}{2} + \frac{bP}{2}[2c \cos A - b]$ or $c^2 = a^2 + b[2c \cos A - b]$.

$= a^2 - b^2 + 2bc \cos A$ or $a^2 = c^2 + b^2 - 2bc \cos A$.

Probability is probably the most problematic branch of mathematics.

A Problem About Probability: A drunkard is taking a random walk. He steps on a line marked with integers $0, 1, 2, ..., n$. For a point x with $0 < x < n$, if he is on x then he can move to $x + 1$ with probability $\frac{1}{2}$ and he can also move to $x - 1$ with the same probability. If he reaches 0 or n then he remains there. Let $p(x)$ denote the probability of reaching n. The function $p(x)$ satisfies (a) $p(0) = 0$ (b) $p(n) = 1$ and (c) $p(x) = \frac{1}{2}[p(x-1) + p(x+1)]$ if $0 < x < n$. (c) follows from the following argument:

He can reach x from $x-1$ or $x+1$, i.e., either from one step backward or from one step forward. His reaching x from position $x-1$ has the same probability as reaching x from position $x + 1$ as he steps in a random fashion. Hence $p(x) = \frac{1}{2}[p(x-1) + p(x+1)]$. We ask the reader to notice that we do not know as yet what the function $p(x)$ is. But whatever $p(x)$ is, we know that it satisfies $p(0) = 0, p(n) = 1$ and $p(x) = \frac{1}{2}[p(x-1) + p(x+1)]$. This is analogous to a differential equation situation where you do not know about $y = f(x)$ but you know the differential equation satisfied by y and also you know the values of y at the boundary points. In fact, the equation satisfied by $p(x)$ is a difference equation with boundary values $p(0) = 0, p(n) = 1$. Also note that the equation $p(x) = \frac{1}{2}[p(x-1) + p(x+1)]$ means that the value of the function at the center x of the interval $(x-1, x+1)$ is the average of the values of p at the boundary points $x-1$, $x+1$ of the interval. Such functions are called harmonic. Condition (c) tells us that the sequence $0 = p(0), p(1), ..., p(n) = 1$ is an A.P whence $p(i) = \frac{i}{n}$.

Sober maths about a drunkard's walk!

Let us see how a little bit of 'electrical jugglery' would help obtain an automatic solution to the problem. The following circuit (*Figure* 13) consists of equal resistors in series fed a unit voltage at the ends. The point O is grounded. Let $v(x)$ denote the voltage at the point x. Clearly $v(0) = 0$ and $v(n) = 1$.

By Ohm's law, $i_{xy} = \frac{v(x)-v(y)}{R}$ if R denotes the resistance across xy and i_{xy} is the current flowing from x to y. Since current flowing into the element $(x, x+1)$ should be equal to the current leaving from the element (x_{i-1}, x), $\frac{v(x)-v(x-1)}{R} = \frac{v(x+1)-v(x)}{R}$ and hence $v(x) = \frac{1}{2}[v(x+1) + v(x-1)]$. Hence $v(x)$ satisfies conditions (a),(b),(c) satisfied by $p(x)$. Hence $v(x) = p(x)$. Thus connecting a voltmeter across O and x, we can obtain the value of $p(x)$.

Simple random walks on an n-dimensional lattice are recurrent if $n = 1$ or 2 and transient for $n > 2$.

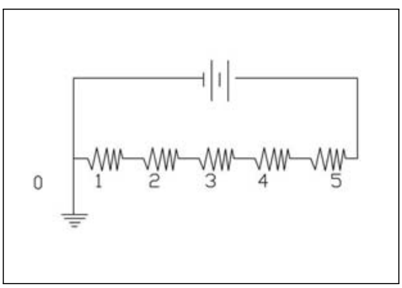


Figure 13. An electrical circuit for the solution of the drunkard's walk problem.

We have illustrated the usefulness of electric circuitry in studying random walks in one dimension. The simplicity of dimension may deceive one into thinking that the use of electric circuits for the study of random walks is silly. The real use of electric networks arises in proving Polya's theorem. In its terse and precise statement the theorem says that – *Simple random walks on an n-dimensional lattice are recurrent if* $n = 1$ *or* 2 *and transient for* $n > 2$. In ordinary terms, the theorem says that *a random walker on the real line or the Euclidean plane will eventually return to the starting point, while, if the dimension of the Euclidean space* $\mathbb{R}^n$ *exceeds* 2*, then the walker may vanish into infinity without ever returning to the starting point*.

We have tried to give an idea of the symbiosis between science and mathematics. Commenting on the purity of 'pure geometry' which ignores all connections to 'physical geometry', philosopher Carl G Hempel says, "Historically speaking, at least, Euclidean geometry has its origin in the generalization, and systematization of certain empirical discoveries which were made in connection with the measurement of areas and volumes, the practice of surveying, and the development of astronomy".

Einstein used non-Euclidean geometry for his theory of relativity, and employed Euclidean geometry in the same theory when considering velocities that are too small in comparison to ordinary velocity of light.

The methods of theoretical physics should be applicable to all those branches of thought in which the essential features are expressed with numbers – Paul Dirac

It might appear enigmatic, even if ingenious to cook up physical situations to model mathematical situations. But this seems to be the need of the hour. More and more of physics is being derived as an abstract mathematical exercise so that the very people involved in the creation of such abstractions are bewildered by their own creations. A classic example is that of Einstein who, inspite of being one of the founders of quantum mechanics, could not come to terms with it. Starting with a physical problem, modeling it mathematically, and then searching for a simple physical analogue for the mathematical model may result in a physical analogue for a physical problem which is more comprehensible! It is something like making a 2nd order differential equation for the study of LCR circuits, and from this pass on to the study of a simple pendulum, and explain the physics of LCR circuits in analogy with that of a simple pendulum. Intuition and abstraction are the two faces of the coin of understanding, and one of them in isolation of the other is sterile. Hence the efforts of people like Mark Levi in inventing physical situations to facilitate the appreciation and understanding of mathematical concepts are laudable.

Acknowledgment

The authors are thankful to the referee for all the valuable suggestions that has led to improvement of the original article.

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