Classroom

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

The Arithmetic Mean – Geometric Mean – Harmonic Mean: Inequalities and a Spectrum of Applications

Prithwijit De
Homi Bhabha Centre for Science Education
Tata Institute of Fundamental Research
Mumbai
Email:
prithwijit@hbcse.tifr.res.in

The Arithmetic Mean – Geometric Mean – Harmonic Mean inequality, AM–GM–HM inequality in short, is one of the fundamental inequalities in Algebra, and it is used extensively in olympiad mathematics to solve many problems. The aim of this article is to acquaint students with the inequality, its proof and various applications.

Before we state the AM–GM–HM inequality, let us define the Arithmetic Mean (AM), Geometric Mean (GM) and Harmonic Mean (HM). Let $a_1, a_2, \ldots, a_n$ be n positive real numbers. The AM (A_n), GM (G_n) and HM (H_n) of these n numbers are defined as:

AM–GM–HM inequality is one of the fundamental inequalities in Algebra.

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}, \tag{1}$$

$$G_n = \sqrt[n]{a_1 a_2 \cdots a_n}, \tag{2}$$

$$H_n = \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}}. \tag{3}$$

The AM–GM–HM inequality states that

$$A_n \geq G_n \geq H_n, \tag{4}$$

Keywords
Arithmetic mean, geometric mean, harmonic mean, Nesbit's inequality, Euler's inequality.

The AM–GM–HM inequality states that $A_n \geq G_n \geq H_n$, and equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

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The result is easy to prove when $n = 2$. In this case we need to show,

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \geq \frac{2}{\frac{1}{a_1} + \frac{1}{a_2}}. \tag{5}$$

Observe that

$$\frac{a_1 + a_2}{2} - \sqrt{a_1 a_2} = \frac{(\sqrt{a_1} - \sqrt{a_2})^2}{2} \geq 0. \tag{6}$$

This proves that $A_2 \geq G_2$. To prove $G_2 \geq H_2$ observe that writing $b_1 = 1/a_1$ and $b_2 = 1/a_2$ reduces the relevant inequality to

$$\frac{b_1 + b_2}{2} \geq \sqrt{b_1 b_2} \tag{7}$$

To prove $G_n \geq H_n$ for any n is same as proving $A_n \geq G_n$ for the reciprocals of the given real numbers.

which we have already proved. This proves the AM–GM–HM inequality for $n = 2$. Note that to prove $G_n \geq H_n$ for any n is same as proving $A_n \geq G_n$ for the reciprocals of the given real numbers. Thus to prove the AM–GM–HM inequality it suffices to prove that $A_n \geq G_n$ for all n. For $n = 3$ the inequality can be proved by using elementary algebra. We have to prove

$$\frac{a + b + c}{3} \geq \sqrt[3]{abc}\ . \tag{8}$$

This is same as proving,

$$(a + b + c)^3 \geq 27abc. \tag{9}$$

But,

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3\,(a + b)\,(b + c)(c + a). \tag{10}$$

Therefore,

$$\begin{aligned}(a + b + c)^3 - 27abc &= (a^3 + b^3 + c^3 - 3abc)\\ &+ 3\{(a + b)\,(b + c)\,(c + a)\\ &- 8abc\}. \end{aligned} \tag{11}$$

We can write,

$$\begin{aligned} a^3+b^3+c^3-3abc &= (a+b)^3+c^3-3ab\,(a+b+c) \\ &= (a+b+c)\,\{(a+b)^2 \\ &- (a+b)\,c+c^2\} \\ &- 3ab\,(a+b+c)\,, \end{aligned} \tag{12}$$

and further simplify it to,

$$a^3+b^3+c^3-3abc = (a+b+c)\,(a^2+b^2+c^2-ab-bc-ca). \tag{13}$$

But,

$$a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}. \tag{14}$$

Thus,

$$a^3+b^3+c^3-3abc = \frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\} \geq 0. \tag{15}$$

Using the AM–GM–HM inequality for $n=2$ we see that,

$$(a+b)(b+c)(c+a) \geq (2\sqrt{ab})\,(2\sqrt{bc})\,(2\sqrt{ca}) = 8abc. \tag{16}$$

Therefore,

$$\begin{aligned} (a+b+c)^3-27abc &= (a^3+b^3+c^3-3abc) \\ &+ 3\{(a+b)\,(b+c)\,(c+a) \\ &- 8abc\} \geq 0\,, \end{aligned} \tag{17}$$

and we have proved $A_3 \geq G_3$. Observe that $A_3 = G_3$ if and only if equality occurs in (15) and (16) which is same as saying $a = b = c$. One may continue in the same spirit to prove the inequality for higher values of n but would realize during the process that the algebra becomes cumbersome and would almost certainly give up the chase. The case $n = 4$ can be dealt with quite elegantly by a clever reduction to the case $n = 2$. How? Observe that,

$$\frac{a_1+a_2+a_3+a_4}{4} = \frac{\dfrac{a_1+a_2}{2}+\dfrac{a_3+a_4}{2}}{2}. \tag{18}$$

Now set $b_1 = \dfrac{a_1 + a_2}{2}$ and $b_2 = \dfrac{a_3 + a_4}{2}$. Then $b_1 \geq \sqrt{a_1 a_2}$, $b_2 \geq \sqrt{a_3 a_4}$ and

$$\frac{\dfrac{a_1 + a_2}{2} + \dfrac{a_3 + a_4}{2}}{2} = \frac{b_1 + b_2}{2} \geq \sqrt{b_1 b_2} = \sqrt[4]{a_1 a_2 a_3 a_4} \ . \tag{19}$$

Hence,

$$\frac{a_1 + a_2 + a_3 + a_4}{4} \geq \sqrt[4]{a_1 a_2 a_3 a_4} \ , \tag{20}$$

with equality if and only if $a_1 = a_2 = a_3 = a_4$. This argument can be easily extended to $n = 8$ and more generally to any n of the form 2^k for some positive integer k. For instance, when $n = 8$, apply $A_4 \geq G_4$ to the set of numbers

$$\left\{\frac{a_1 + a_2}{2}, \frac{a_3 + a_4}{2}, \frac{a_5 + a_6}{2}, \frac{a_7 + a_8}{2}\right\}$$

followed by $A_2 \geq G_2$ to the sets of numbers $\{a_1, a_2\}, \{a_3, a_4\}$, $\{a_5, a_6\}, \{a_7, a_8\}$. Thus for $n = 2^k$,

$$\frac{a_1 + a_2 + \cdots + a_{2^k}}{2^k} \geq \sqrt[2^k]{a_1 a_2 \cdots a_{2^k}} \ . \tag{21}$$

If n is not a power of two then we can find a unique k such that $2^k < n < 2^{k+1}$ and we observe that

$$\begin{aligned} A_n &= \frac{a_1 + a_2 + \cdots + a_n + \overbrace{(A_n + \cdots + A_n)}^{(2^{k+1} - n) \text{ times}}}{2^{k+1}} \\ &\geq \sqrt[2^{k+1}]{a_1 a_2 \cdots a_n . (A_n)^{2^{k+1} - n}} \ . \end{aligned} \tag{22}$$

This can be simplified further to obtain,

$$A_n^n \geq a_1 a_2 \cdots a_n \ , \tag{23}$$

and this is same as $A_n \geq G_n$. The idea of this proof is due to the great French mathematician Augustin-Louis Cauchy (1789–1857). There is another way to prove $A_n \geq G_n$ for all positive integers n. Let,

$$x_i = \frac{a_i}{\sqrt[n]{a_1 a_2 \cdots a_n}} \ , \tag{24}$$

for $i = 1, 2, \ldots, n$. Then $x_1 x_2 \cdots x_n = 1$ and to prove $A_n \geq G_n$ we need to show that,

$$x_1 + x_2 + \cdots + x_n \geq n\,, \tag{25}$$

subject to $x_1 x_2 \cdots x_n = 1$. When $n = 2$ it boils down to showing $x_1 + x_2 \geq 2$ if $x_1 x_2 = 1$. This follows easily because

$$x_1 + x_2 - 2 = (\sqrt{x_1})^2 + (\sqrt{x_2})^2 - 2\sqrt{x_1 x_2} = (\sqrt{x_1} - \sqrt{x_2})^2 \geq 0. \tag{26}$$

AM–GM–HM inequality is used extensively in olympiad mathematics to solve many problems.

Assume that we have shown that for $m \geq 3$,

$$x_1 + x_2 + \cdots + x_{m-1} \geq m - 1\,, \tag{27}$$

if $x_1 x_2 \cdots x_{m-1} = 1$. Now $x_1 x_2 \cdots x_m = 1$. Without loss of generality let $x_1 \leq 1 \leq x_2$. Thus,

$$(x_2 - 1)(1 - x_1) \geq 0\,, \tag{28}$$

implying

$$x_1 + x_2 \geq 1 + x_1 x_2. \tag{29}$$

Observe that $x_1 x_2 \cdots x_m = 1 = y_1 y_2 \cdots y_{m-1}$ where $y_1 = x_1 x_2$ and $y_i = x_{i+1}$ for $2 \leq i \leq m-1$. Therefore,

$$y_1 + y_2 + \cdots + y_{m-1} \geq m - 1. \tag{30}$$

But $y_2 + y_3 + \cdots + y_{m-1} = x_3 + x_4 + \cdots + x_m$ and $1 + y_1 = 1 + x_1 x_2 \leq x_1 + x_2$. Thus,

$$x_1 + x_2 + \cdots + x_m \geq m\,, \tag{31}$$

and by induction it follows that $x_1 + x_2 + \cdots + x_n \geq n$ for every positive integer $n \geq 2$ if $x_1 x_2 \cdots x_n = 1$.

Applications

Having presented two proofs of the AM–GM–HM inequality, let us look at some applications.

1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Let $b_1, b_2, \ldots, b_n$ be a permutation of $a_1, a_2, \ldots, a_n$. Then,

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n.$$

In order to prove this just note that $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_n$. Thus,

$$\frac{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}}{n} \geq \sqrt[n]{\frac{a_1 a_2 \ldots a_n}{b_1 b_2 \ldots b_n}} = 1. \tag{32}$$

2. (Nesbit's Inequality)

Let a, b, c be positive real numbers. Then,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

and equality occurs if and only if $a = b = c$.

Let $P = \dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$. Then,

$$P + 3 = (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right). \tag{33}$$

Now $A_3 \geq H_3$ implies,

$$\frac{\frac{a+b}{2} + \frac{b+c}{2} + \frac{c+a}{2}}{3} \geq \frac{3}{\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a}}. \tag{34}$$

Therefore $P + 3 \geq \dfrac{9}{2}$ and hence $P \geq \dfrac{3}{2}$. Equality occurs if and only if $A_3 = H_3$ which occurs if and only if $\dfrac{a+b}{2} = \dfrac{b+c}{2} = \dfrac{c+a}{2}$, i.e., $a = b = c$.

3. Let a, b, c be real numbers greater than or equal to 1. Then,

$$\frac{a^3+2}{b^2-b+1} + \frac{b^3+2}{c^2-c+1} + \frac{c^3+2}{a^2-a+1} \geq 9.$$

To see this, observe that $a^3+2-3\,(a^2-a+1) = a^3-3a^2+3a-1 = (a-1)^3 \geq 0$, whence $\dfrac{a^3+2}{a^2-a+1} \geq 3$. Since every term in the left hand side of the inequality is positive we have,

$$\frac{a^3+2}{b^2-b+1}+\frac{b^3+2}{c^2-c+1}+\frac{c^3+2}{a^2-a+1}$$
$$\geq 3\left(\prod \frac{(a^3+2)(b^3+2)(c^3+2)}{(a^2-a+1)(b^2-b+1)(c^2-c+1)}\right)^{1/3} = 9. \quad (35)$$

4. Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Then, we have the inequality,

$$\frac{1}{(a-1)(b-1)(c-1)}+\frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$

For this, observe that $a>1, b>1, c>1$ and $a-1 = a\left(\frac{1}{b}+\frac{1}{c}\right) \geq \frac{2a}{\sqrt{bc}}$ (by AM–GM inequality). Similarly, we get $b-1 \geq \frac{2b}{\sqrt{ca}}$ and $c-1 \geq \frac{2c}{\sqrt{ab}}$. Multiplying these and taking the reciprocal we obtain,

$$\frac{1}{(a-1)(b-1)(c-1)} \leq \frac{1}{8}. \quad (36)$$

Next observe that $\frac{a+1}{a-1} = 1+\frac{2}{a-1} \geq \frac{2\sqrt{2}}{\sqrt{a-1}}$ whence, $a+1 \geq 2\sqrt{2(a-1)}$. Similarly we obtain $b+1 \geq 2\sqrt{2(b-1)}$ and $c+1 \geq 2\sqrt{2(c-1)}$. Multiplying these yields

$$(a+1)(b+1)(c+1) \geq 16\sqrt{2(a-1)(b-1)(c-1)} \geq 16\sqrt{2.8} = 64.$$

Therefore,

$$\frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{8}. \quad (37)$$

By adding (36) and (37) we get,

$$\frac{1}{(a-1)(b-1)(c-1)}+\frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$

5. (Euler's Inequality)

Let R be the circumradius and r be the inradius of a triangle. Then $R \geq 2r$ with equality if and only if the triangle is equilateral.

This is perhaps the most elegant geometric inequality that one comes across in school level mathematics. There is a geometric proof of this result. Here, we shall present an algebraic proof.

Let ABC be the given triangle. Denote by a, b, and c the lengths of the sides BC, CA, and AB respectively. Let $s = \dfrac{a+b+c}{2}$, where s is the semi-perimeter, and let Δ denote the area of ABC. Observe that $s-a$, $s-b$ and $s-c$ are positive quantities. Recall that $r = \dfrac{\Delta}{s}$ and $R = \dfrac{abc}{4\Delta}$. Therefore,

$$\frac{R}{r} = \frac{abcs}{4\Delta^2} = \frac{abc}{4\,(s-a)\,(s-b)\,(s-c)}. \tag{38}$$

Now observe that,

$$a = 2s - b - c = (s-b) + (s-c) \geq 2\sqrt{(s-b)\,(s-c)}, \tag{39}$$

$$b = 2s - c - a = (s-c) + (s-a) \geq 2\sqrt{(s-c)\,(s-a)}, \tag{40}$$

and,

$$c = 2s - a - b = (s-a) + (s-b) \geq 2\sqrt{(s-a)\,(s-b)}. \tag{41}$$

Multiplying these three inequalities leads to,

$$abc \geq 8(s-a)\,(s-b)\,(s-c) \tag{42}$$

whence,

$$\frac{R}{r} = \frac{abc}{4(s-a)\,(s-b)\,(s-c)} \geq 2. \tag{43}$$

Equality occurs if and only if $s-a = s-b = s-c$, that is, $a = b = c$, which is same as saying that the triangle is equilateral.

Solving Some Equations Using the Inequalities

AM–GM–HM inequality can be used to solve algebraic equations.

In all the examples discussed so far, we applied the AM–GM–HM inequality to show that one algebraic expression is at least as large as another algebraic expression under certain conditions on the variables involved. It can also be used to solve algebraic equations. Here are some examples.

1. Suppose we wish to find all quadruples a, b, c, d of positive real numbers that are solutions to the system of equations

$$a+b+c+d=4,$$
$$\left(\frac{1}{a^{12}}+\frac{1}{b^{12}}+\frac{1}{c^{12}}+\frac{1}{d^{12}}\right)(1+3abcd)=16.$$

By $A_4 \geq G_4$ we get,

$$1=\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}\,, \tag{44}$$

$$\frac{\frac{1}{a^{12}}+\frac{1}{b^{12}}+\frac{1}{c^{12}}+\frac{1}{d^{12}}}{4} \geq \sqrt[4]{(abcd)^{-12}}\,, \tag{45}$$

Multiplying both sides of (45) by $(1+3abcd)$ and simplifying we obtain,

$$\left(\frac{1}{a^{12}}+\frac{1}{b^{12}}+\frac{1}{c^{12}}+\frac{1}{d^{12}}\right)(1+3abcd) \geq \frac{4(1+3abcd)}{(abcd)^3}. \tag{46}$$

Let $t=abcd$. Then $16 \geq \frac{4(1+3t)}{t^3}$. This implies $4t^3-3t-1 \geq 0$. But $4t^3-3t-1=(t-1)(2t+1)^2$. Therefore $(t-1)(2t+1)^2 \geq 0$. Hence $t \geq 1$. But earlier we obtained $t \leq 1$. Therefore $t=1$.

So now we have,

$$\frac{a+b+c+d}{4}=\sqrt[4]{abcd}\,,$$

that is, $A_4=G_4$. Hence $a=b=c=d=\frac{a+b+c+d}{4}=1$. It is easy to see that $(a,b,c,d)=(1,1,1,1)$ indeed satisfies the given system of equations.

2. Let us solve the simultaneous system of equations for real values of x, y and z.

$$x = \frac{2y^2}{1+y^2}; \; y = \frac{2z^2}{1+z^2}; \; z = \frac{2x^2}{1+x^2}.$$

Observe that $x = y = z = 0$ is a solution. If x, y, z are different from zero then they must be positive (Why?). Now observe that for any positive real number t,

$$\frac{1}{2}\left(t + \frac{1}{t}\right) \geq \sqrt{t.\frac{1}{t}} = 1. \tag{47}$$

Therefore,

$$x = \frac{y}{\frac{1}{2}\left(y + \frac{1}{y}\right)} \leq y, \tag{48}$$

$$y = \frac{z}{\frac{1}{2}\left(z + \frac{1}{z}\right)} \leq z, \tag{49}$$

and,

$$z = \frac{x}{\frac{1}{2}\left(x + \frac{1}{x}\right)} \leq x. \tag{50}$$

Thus $x \leq y \leq z \leq x$. Hence $x = y = z$ and the common value is 1. Thus the solutions are $(x, y, z) = (0, 0, 0), (1, 1, 1)$. It is easy to see that these triples indeed satisfy the given system.

Optimization Problems

Some optimization questions can also be solved using the AM–GM–HM inequalities.

1. Among all triangles with a fixed perimeter, let us find which triangle has the largest area. Let us also determine the maximum area.

Let p be the fixed perimeter. Let a, b and c be the sides of a triangle whose perimeter is p. Let $s = p/2$. The area of the triangle is

$$\Delta(a,b,c) = \sqrt{s(s-a)(s-b)(s-c)}\ . \tag{51}$$

Some optimization questions can also be solved using the AM–GM–HM inequalities.

Note that the area is a function of the side-lengths a, b and c. Now,

$$\frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}, \tag{52}$$

with equality if and only if $s-a = s-b = s-c$, that is, $a = b = c$. Therefore,

$$\Delta(a,b,c) = \sqrt{s(s-a)(s-b)(s-c)} \leq \frac{s^2}{3\sqrt{3}} = \frac{p^2}{12\sqrt{3}}. \tag{53}$$

Thus the maximum value of the area is $\dfrac{p^2}{12\sqrt{3}}$ and it is attained when the triangle is equilateral.

2. Let a sector with central angle θ be removed from a circle of unit radius and rolled into a right circular cone. What is the value of θ for which the volume of the cone thus formed is maximum?

Let r and h be respectively the radius of the base and the height of the cone. Then,

$$2\pi r = \theta\,, \tag{54}$$

and,

$$\pi r\sqrt{r^2+h^2} = \frac{\theta}{2}\ . \tag{55}$$

Solving for h to obtain,

$$h = \sqrt{1-r^2} = \sqrt{1-\left(\frac{\theta}{2\pi}\right)^2}. \tag{56}$$

Therefore the volume of the cone is,

$$V = \frac{\pi r^2 h}{3} = \frac{\pi}{3}\cdot\left(\frac{\theta}{2\pi}\right)^2\sqrt{1-\left(\frac{\theta}{2\pi}\right)^2}\ . \tag{57}$$

AM–GM–HM inequalities can be applied to a system of quadratic equations.

Let $t = \left(\frac{\theta}{2\pi}\right)^2$. Observe that $0 < t < 1$ and,

$$\frac{\frac{t}{2} + \frac{t}{2} + (1-t)}{3} \geq \sqrt[3]{\frac{t}{2} \cdot \frac{t}{2} \cdot (1-t)} \tag{58}$$

whence,

$$t\sqrt{1-t} \leq \frac{2}{3\sqrt{3}}. \tag{59}$$

Equality is attained if and only if $\frac{t}{2} = 1 - t$, that is, $t = \frac{2}{3}$ and this leads to $\theta = 2\pi\sqrt{\frac{2}{3}}$. Therefore the volume of the cone is maximised when the central angle is equal to $2\pi\sqrt{\frac{2}{3}} = 294°$ approximately.

Here is an application of the AM–GM–HM inequalities to a system of quadratic equations.

Suppose that the three equations $ax^2-2bx+c = 0$, $bx^2-2cx+a = 0$ and $cx^2 - 2ax + b = 0$ all have only positive roots. Show that $a = b = c$.

Let α_1, β_1 be the roots of $ax^2 - 2bx + c = 0$, α_2, β_2 be the roots of $bx^2 - 2cx + a = 0$ and α_3, β_3 be the roots of $cx^2 - 2ax + b = 0$. Then,

$$\alpha_1 + \beta_1 = \frac{2b}{a}; \;\; \alpha_1\beta_1 = \frac{c}{a}, \tag{60}$$

$$\alpha_2 + \beta_2 = \frac{2c}{b}; \;\; \alpha_2\beta_2 = \frac{a}{b}, \tag{61}$$

$$\alpha_3 + \beta_3 = \frac{2a}{c}; \;\; \alpha_3\beta_3 = \frac{b}{c}. \tag{62}$$

The roots are positive. By $A_2 \geq G_2$ we obtain,

$$\frac{b}{a} = \frac{\alpha_1 + \beta_1}{2} \geq \sqrt{\alpha_1\beta_1} = \sqrt{\frac{c}{a}}, \tag{63}$$

$$\frac{c}{b} = \frac{\alpha_2 + \beta_2}{2} \geq \sqrt{\alpha_2\beta_2} = \sqrt{\frac{a}{b}}, \tag{64}$$

$$\frac{a}{c} = \frac{\alpha_3 + \beta_3}{2} \geq \sqrt{\alpha_3\beta_3} = \sqrt{\frac{b}{c}}, \tag{65}$$

which are equivalent to $b^2 \geq ca$, $c^2 \geq ab$ and $a^2 \geq bc$. Suppose on the contrary, either a, b, c are distinct or any two of them are equal but are different from the third. If a, b, c are distinct then as the inequalities are symmetric with respect to a, b, c, we may assume without loss of generality that $a > b > c$. In that case $c^2 < ab$ and we have a contradiction. If $a = b > c$ then also $c^2 < ab$ and we have a contradiction. By symmetry $a > b = c$ and $c = a > b$ will lead to contradictions too. Therefore $a = b = c$.

Weighted AM–GM–HM Inequalities

Let us point out to the reader that for the AM–GM inequality to hold, the quantities $a_1, a_2, \ldots, a_n$ may assumed to be non-negative but they have to be positive for the AM–HM inequality or the GM–HM inequality to be valid. The AM–GM–HM inequality can be generailzed to the weighted AM–GM–HM inequality by assigning weights to the quantities. We state the inequality without proof.

Let $a_1, a_2, \ldots, a_n$ and $w_1, w_2, \ldots, w_n$ be positive real numbers. Then,

$$\frac{\sum\limits_{i=1}^{n} w_i a_i}{\sum\limits_{i=1}^{n} w_i} \geq (a_1^{w_1} \cdot a_2^{w_2} \cdots a_n^{w_n})^{\frac{1}{\sum_{i=1}^{n} w_i}} \geq \frac{\sum\limits_{i=1}^{n} w_i}{\sum\limits_{i=1}^{n} \frac{w_i}{a_i}}.$$

Here is an application.

Let a, b, c be positive real numbers. Prove that

$$\frac{1}{10a+11b+11c}+\frac{1}{11a+10b+11c}+\frac{1}{11a+11b+10c}\leq \frac{1}{32a}+\frac{1}{32b}+\frac{1}{32c}.$$

By the weighted AM–HM inequality we have,

$$\frac{32}{10a+11b+11c}\leq\frac{10/a+11/b+11/c}{32},$$
$$\frac{32}{11a+10b+11c}\leq\frac{11/a+10/b+11/c}{32},$$
$$\frac{32}{11a+11b+10c}\leq\frac{11/a+11/b+10/c}{32}.$$

Upon adding the above inequalities and rearranging terms we obtain,

$$\frac{1}{10a+11b+11c}+\frac{1}{11a+10b+11c}+\frac{1}{11a+11b+10c}\leq \frac{1}{32a}+\frac{1}{32b}+\frac{1}{32c}.$$

We conclude this article by leaving some problems for the student.

Problems

1. Let $a_1, a_2, \ldots, a_n$ be n positive real numbers whose product is 1. Prove that

$$(1+a_1)(1+a_2)\cdots(1+a_n)\geq 2^n.$$

2. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\frac{a(b^2+3)}{3c^2+1}+\frac{b(c^2+3)}{3a^2+1}+\frac{c(a^2+3)}{3b^2+1}\geq 3.$$

3. Let $0=a_0<a_1<\cdots<a_n<a_{n+1}=1$ be such that $a_1+a_2+\cdots+a_n=1$. Prove that

$$\frac{a_1}{a_2-a_0}+\frac{a_2}{a_3-a_1}+\cdots+\frac{a_n}{a_{n+1}-a_{n-1}}\geq\frac{1}{a_n}.$$

4. Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $x_1 + x_2 + \cdots + x_n = 1$. Then show that

$$\sum_{i=1}^{n} \frac{x_i}{2 - x_i} \geq \frac{n}{2n-1}.$$

5. If $a, b, c \in (0, 1)$ satisfy $a + b + c = 2$, prove that
$\dfrac{abc}{(1-a)(1-b)(1-c)} \geq 8.$

6. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{2a+b+c} + \frac{b}{2b+c+a} + \frac{c}{2c+a+b} \leq \frac{a}{2b+2c} + \frac{b}{2c+2a} + \frac{c}{2a+2b}.$$

7. Let a, b, c be positive real numbers such that $ab+bc+ca = 3$. Prove that

$$(3a^2+2)\left(\frac{a^3+b^3}{a^2+ab+b^2}\right) + (3b^2+2)\left(\frac{b^3+c^3}{b^2+bc+c^2}\right) + (3c^2+2)\left(\frac{c^3+a^3}{c^2+ca+a^2}\right) \geq 10abc.$$

8. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{(\sqrt{a}+\sqrt{b})^4}{a+b} + \frac{(\sqrt{b}+\sqrt{c})^4}{b+c} + \frac{(\sqrt{c}+\sqrt{a})^4}{c+a} \geq 24.$$

9. Let h_a, h_b and h_c be the altitudes and r the inradius of a triangle. Prove that

$$\frac{h_a-2r}{h_a+2r} + \frac{h_b-2r}{h_b+2r} + \frac{h_c-2r}{h_c+2r} \geq \frac{3}{5}.$$

10. Let a, b, c be positive numbers and $abc = 8$. Prove that

$$\frac{a^4+b^4}{c^3} + \frac{b^4+c^4}{a^3} + \frac{c^4+a^4}{b^3} \geq 64\left(\frac{1}{a^5} + \frac{1}{b^5} + \frac{1}{c^5}\right) + 6$$

Suggested Reading

[1] B J Venkatachala, *Inequalities - An Approach Through Problems*, Hindustan Book Agency.