Geometry of Spin: Clifford Algebraic Approach

Rukhsan-Ul-Haq

Spin is a fundamental degree of freedom of matter and radiation. In quantum theory, spin is represented by Pauli matrices. Then the various algebraic properties of Pauli matrices are studied as properties of matrix algebra. What has been shown in this article is that Pauli matrices are a representation of Clifford algebra of spin and hence all the properties of Pauli matrices follow from the underlying algebra. Clifford algebraic approach provides a geometrical and hence intuitive way to understand quantum theory of spin, and is a natural formalism to study spin. Clifford algebraic formalism has lot of applications in every field where spin plays an important role.

Rukhsan-Ul-Haq is a student of JNCASR, Bengaluru and is interested in the area of topological order in quantum many body systems and their applications in topological quantum computation. He is also interested in bringing mathematics to a broader audience.

Introduction

Clifford algebras were discovered twice in physics. First, when Pauli introduced spin in quantum mechanics and secondly when Dirac introduced the relativistic equation for the electron. Both of them were using the same algebra albeit different representations of it. The algebra they were using was known to mathematicians before them, and is called 'Clifford algebra'. Pauli matrices and Dirac matrices are representations of the Clifford algebra. Pauli matrices define two-dimensional representation of Euclidean signature (all of them square to unity) while the Dirac matrices define four-dimensional representation of Lorentz signature (three of them square to +1 and the other one squares to –1). Physicists usually deal with matrices (representations) rather than with their abstract algebra and therefore the approach is very algebraic. Hence when beginners get introduced to these matrices and their algebra, they get lost in various algebraic identities which are derived from the properties of the matrices rather than from the underlying algebra. This would have put the mathemati-

Keywords
Geometry of spin, Pauli matrices, Clifford algebra.

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cians who introduced this algebra at loss because as the name indicates, this algebra is essentially geometric. In fact as we will see below, Clifford algebras are defined for metric spaces, where one can define the notion of length and hence geometry. In this article, we will take the Clifford algebraic approach to understand quantum mechanics of spin. Not only will we get a more geometrical and physical understanding of spin, but more importantly, we will also uncover many mathematical properties of spin which are not transparent in standard approach. One very important feature which is not quite visible in the matrix representations is the graded (Z_2 grading[1]) nature of the algebra of spin. All the generators of the algebra do not have the same rank (in the sense of tensor algebra). We can have scalars, vectors, and bivectors which are of different ranks in the same algebra. Bivectors and trivectors are of rank two and three respectively, and represent two different geometrical entities. We will see that bivector is very important in understanding the rotational properties of spin because they generate the Lie algebra of rotation, and hence one can see that Lie algebra is already a subalgebra of Clifford algebra. The spaces on which Clifford algebras operate are called 'Spinor spaces', and their elements are called 'Spinors'. To understand Clifford algebra, we will start with familiar algebras like complex numbers and quaternions which are also known as hypercomplex numbers. It will help us understand the Clifford algebra of three-dimensional Euclidean space which will be shown to be related to the algebra of spin, and hence we will understand the geometry of spin which is essentially the quantum mechanical degree of freedom of both matter and radiation.

[1]Clifford algebra has two subalgebras which are called even and odd algebras, and due to which there is Z_2 (even-odd) grading of the elements of the algebra.

The rest of the article is organized this way. We will first present the standard formalism of quantum theory of spin in section 1. In the next sections we introduce Clifford algebra and describe the Clifford algebra of Euclidean plane Cl_2. We also show that algebra of complex numbers is related to this algebra. This way we get new insights into the geometry of complex numbers. We also talk about quaternions before going into details of Clifford algebra of three-dimensional Euclidean space Cl_3 in section 4.

And finally, we show how the algebra of spin is related to Cl_3 by showing that Pauli matrices give representations of this algebra. Hence, all the properties of spin are explained using Clifford algebra and we explore the geometry of quantum spin. Then we go on to show that all the algebraic properties and identities of Pauli matrices follow from Cl_3. Using this algebra we prove two standard identities of Pauli matrices without invoking the matrix algebra. In the last section we summarize the conclusions and results.

1. Pauli Spin Algebra

We will first present Pauli spin algebra as prescribed by standard textbooks [1, 2]. This algebra was introduced to explain the electron spin as observed in Stern–Gerlach experiment where silver atoms get polarized in only two directions. One writes spin operators in terms of Pauli matrices,

$$S_x = \frac{h}{2}\sigma_x, \quad S_y = \frac{h}{2}\sigma_y, \quad S_z = \frac{h}{2}\sigma_z, \tag{1}$$

where σ_x, σ_y, and σ_z are Pauli matrices and satisfy the following algebra.

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \tag{2}$$

$$\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z \tag{3}$$

$$\sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x \tag{4}$$

$$\sigma_z\sigma_x = -\sigma_x\sigma_z = i\sigma_y \tag{5}$$

$$\sigma_x\sigma_y\sigma_z = i \tag{6}$$

$$Tr\sigma_x = Tr\sigma_y = Tr\sigma_z = 0 \tag{7}$$

$$det\sigma_x = det\sigma_y = det\sigma_z = -1\,. \tag{8}$$

2. What is Clifford Algebra?

Clifford algebra is defined for metric spaces in which we have the notion of length or distance. The spaces which we are familiar with in physics are all metric spaces. The well-known example is

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Euclidean space. Clifford algebra is tied to geometry, and hence is also called geometric algebra [3]. In this algebra we already have synthesis of geometry and algebra. Clifford algebra has central importance both in physics and mathematics. In physics, Clifford algebra enters many places but becomes specially important when we deal with fermions (electrons, etc.,). In mathematics, Clifford algebra is used to construct the spinorial representations of Lie algebras, and also to study spinorial properties of manifolds [4, 5, 6]. Though there are many different ways of defining Clifford algebra [3, 6, 7, 8], we will stick to a definition which allows comparison with Pauli Spin matrices. We will use the definition of Clifford algebra in terms of generators which are geometrical entities. The generators of the Clifford algebra satisfies,

$$e_i^2 = 1, \quad e_i e_j = -e_j e_i \,. \tag{9}$$

More compactly, above properties can be written as:

$$\{e_i, e_j\} = 2\delta_{ij}\,, \tag{10}$$

where we have taken anti-commutator of two generators. e_i and e_j are orthonormal vectors of the given space.

To understand these relations, we will introduce 'Clifford product'. Let e_1 and e_2 be two vectors, then Clifford product for these two vectors is given as:

$$e_1 e_2 = e_1.e_2 + e_1 \wedge e_2\,, \tag{11}$$

where the first term is the scalar product and second term is the exterior product. Clifford product is written as juxtaposition of two vectors. Exterior product was introduced by Grassmann [7], and is widely used in differential geometry to define differential forms [8, 9].

It should be noted that exterior product is different from cross product which is specific to three-dimensional space, and hence is called the 'accident of the three-dimensional space'.

The thing to be noted over here is that the exterior product of two vectors is a new geometrical quantity called bivector which is different from vector because it has a higher rank. Geometrically, bivector represents an oriented area covered by a plane formed by two vectors. Exterior product can be used to get higher rank multivectors. It should be noted that exterior product is different from cross product which is specific to three dimensional space and hence is called the 'accident of the three-dimensional space'. Cross product returns a vector and needs metric for its definition while exterior product returns a higher rank multivector and does not need any metric for its definition. Exterior product can be generalized to spaces of arbitrary dimension. However, in the three-dimensional space, cross product is related to exterior product through a 'Hodge duality relation'[2] which uses the duality operator or the volume operator or the pseudoscalar. It relates vectors to bivectors in 3D or geometrically it relates lines to planes.

[2]Hodge duality is the relation between vectors and bivectors. In $Cl(3,0)$, trivector is the duality operator which maps an element to its Hodge dual.

Scalar product is symmetric:

$$e_1.e_2 = e_2.e_1 \,, \tag{12}$$

and needs metric for its definition. Exterior product is anticommutative and does not need any metric for its definition.

$$e_1 \wedge e_2 = -e_2 \wedge e_1 \,. \tag{13}$$

The anticommutativity can be understood geometrically. Since bivector represents an oriented area, changing the order of vectors changes the orientation and hence the minus sign.

3. Complex Numbers

The real numbers generate the simplest Clifford algebra. However, we will discuss about the Clifford algebra of complex numbers because it is in the complex plane where we get first glimpse of the magic of Clifford algebra. Along the way we will understand the geometry of imaginary unit. Clifford algebra of complex plane is called Cl_2. The set of its generators is $(1, e_1, e_2, e_{12})$. There is a scalar element, two vector generators, and one bivector generator.

$$e_1^2 = e_2^2 = 1 \tag{14}$$

$$e_{12}^2 = -1 \tag{15}$$

$$e_1e_{12} = e_2\,, \quad e_{12}e_1 = -e_2 \tag{16}$$

$$e_2e_{12} = -e_1\,, \quad e_{12}e_2 = e_1 \tag{17}$$

e_{12} is a very special element of the algebra. It squares to -1 like imaginary unit, and anticommutes with vector generators and hence all the vector elements of the algebra. Cl_2 can be written as direct sum of algebras generated by different generators.

$$Cl_2 = R \oplus R^2 \oplus \bigwedge^2 R^2\,, \tag{18}$$

where R denotes real number field, R^2 denotes two-dimensional Euclidean plane, and $\bigwedge^2 R^2$ denotes space of bivectors which in this case is one-dimensional. Cl_2 has one subalgebra which we get from direct sum of R and $\bigwedge^2 R^2$ and this algebra is isomorphic to C.

$$C \cong R \oplus \bigwedge^2 R^2\,. \tag{19}$$

The mystery of negative sign in the algebra of complex numbers gets unfolded in their geometry.

That means, any complex number Z can be written as sum of scalar generator and bivector multiplied by the real number coefficients – $Z = x + ye_{12}$. Unit bivector acts as imaginary unit! So in this representaion of complex numbers, geometry of imaginay unit becomes explicit. Imaginary unit is the oriented area in Euclidean plane. In this way we can easily understand why imaginary unit squares to -1, because if we change the order of the vectors we will get negative sign. The mystery of negative sign in the algebra of complex numbers gets unfolded in their geometry. Imaginary unit plays another important role in rotations in Euclidean plane, and is also related to an important symmetry in physics. We will not go into details of this aspect of complex numbers but will explain it briefly here. When we multiply any real number which lies on real axis by the imaginay unit, it gets rotated to the perpendicular axis. So imaginary unit actually rotates by 90 degress which is not very clear if we take imaginary numbers just as a number. This rotational property can be explained only when we take the imaginary unit as a bivector which sends e_1 to one axis and e_2 to the perpendicular axis. The connection of imaginary unit to rotations is another beautiful chapter in Clifford algebraic and spinorial properties of complex numbers, which has important applications in physics. We summarize this section by emphasizing on following points.

1. We showed that algebra of complex numbers is the subalgebra of Clifford algebra of Euclidean plane.

2. Using the Clifford algebraic approach we understood the geometrical significance of imaginary unit, and we also saw that the origin of negative sign follows from the geometry of imaginary unit.

3. We also touched on the rotational properties of the imaginary unit.

4. Clifford Algebra of Euclidean Space

As we saw, complex numbers have a very beautiful algebra. So it becomes natural to ask whether this algebra can be generalized to higher dimensions, three dimensions being an interesting case. This work was carried out by Sir Rowan William Hamilton who is known for Hamiltonian dynamics and proposing the first very important generalization of complex numbers. He called his numbers as 'Quaternions' [7, 11, 12]. While he was trying to write an algebra similar to complex numbers for three dimensions, he realized that it is only for four dimensions that we can have such an algebra. In the process he introduced quaternion algebra which has three imaginary units. The birth of quaternions is historically documented. On 16th October 1843, while crossing the Brougham bridge with his wife, Hamilton got a flash of creativity, and instantly sat down and inscribed the quaternion al-

Sir Rowan William Hamilton who is known for Hamiltonian dynamics and proposing the first very important generalization of complex numbers also proposed the idea of quaternions.

gebra on the bridge.

$$i^2 = j^2 = k^2 = -1\,, \tag{20}$$
$$ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j\,, \tag{21}$$
$$ijk = -1\,. \tag{22}$$

We refer the curious readers, who want to explore more of this beautiful subject to two very nice expositions on hypercomplex numbers for [7, 13].

Now we are ready to study Clifford algebra of Euclidean space Cl_3. It is generated by the following elements:

$$1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\,. \tag{23}$$

It is an eight-dimensional algebra because it has eight generators. In addition to vectors, it also has bivectors and a trivector. Vector generators are orthonormal. All these generators are geometrical entities. This is typical of Clifford algebra where we have synthesis of algebra and geometry.

$$e_1^2 = e_2^2 = e_3^2 = 1 \tag{24}$$
$$e_{ij} = e_i e_j = e_i.e_j + e_i \wedge e_j = e_i \wedge e_j \tag{25}$$
$$e_{ij}^2 = (e_i e_j)\,(e_i e_j) = -1 \tag{26}$$
$$e_{123} = e_1 e_2 e_3 = e_1 \wedge e_2 \wedge e_3 \tag{27}$$
$$e_{123}^2 = (e_1 e_2 e_3)\,(e_1 e_2 e_3) = -1\,. \tag{28}$$

Here,

$$1 \qquad Scalar \tag{29}$$
$$e_1, e_2, e_3 \qquad Vectors \tag{30}$$
$$e_{12}, e_{13}, e_{23} \qquad Bivectors \tag{31}$$
$$e_{123} \qquad Trivecor \tag{32}$$

Pauli spin matrices are isomorphic to Cl_3, and give its matrix representation. The mapping can be written as:

$\underline{Cl_3}$	$\underline{Mat\,(2,C)}$	
1	I	(33)
e_1	σ_1	(34)
e_2	σ_2	(35)
e_3	σ_3	(36)

Since Pauli spin matrices are a representation of Cl_3, all the algebraic properties of Pauli spin matrices follow from the underlying Clifford algebra. The main aim of this article is to bring forth this underlying algebraic structure of Pauli matrices which makes their algebra more intuitive, and gives it a geometric flavor. One can easily see that all the algebraic relations of Pauli matrices follow from the geometrical properties of generators of the Clifford algebra. This also motivates the connection of spin to the geometry of Euclidean space [10]. One important result which we will show is that Pauli matrices also form a representation of Lie algebra of rotation. This Lie algebra exists as subalgebra in Cl_3 and is generated by bivectors.

4.1 *Pauli Spin Algebra from Clifford Algebra*

In this section, we will show that all the properties of Pauli matrices given in (2–8) follows from the underlying Clifford algebra. One need not refer to matrix algebra to prove these identities. So one can see that (2–3) follows from the definition of Clifford algebra. Pauli matrices square to one and anticommute with each other. Now in Pauli formalism, one takes these relations as definition of the algebra, but in Clifford algebra one can understand them geometrically. In Pauli formalism one does not understand why two matrices should anticommute, but in the Clifford algebra, the product of two vectors gives a bivector. Bivector changes its sign when order of the vectors is changed as it has an orientation associated with it. Hence one can see that anticommuting property of Pauli matrices follows from the geometry. One does not have to take anticommutation as an axiom, rather one can see it directly as a geometric property of bivectors which themselves come form the exterior product between two orthogonal vectors. Along the same lines one can also understand the Lie algebra generated by Pauli matrices.

4.2 *Algebra of Bivectors: Lie Algebra of Pauli Matrices*

Bivectors are very important part of the Clifford algebra because they generate rotations, and hence give representations of SO (3) group. This representation is called 'spinor represenation', and this group is called 'spin group', and both of them are ismomorphic to SU(2) group of Pauli matrices.

$$e_1e_2 - e_2e_1 = [e_1, e_2] = e_{123}e_3 \tag{37}$$

$$e_2e_3 - e_3e_2 = [e_2, e_3] = 2e_{123}e_1 \tag{38}$$

$$e_3e_1 - e_1e_3 = [e_3, e_1] = 2e_{123}e_2\,. \tag{39}$$

One can write them compactly,

$$[e_j, e_k] = i\sum_l \epsilon_{jkl}e_l\,, \tag{40}$$

where we have replaced e_{123} by i. We can once again see that this Lie algebra is generated by bivectors. What is not evident in Pauli formalism is the role of i factor in the equation. In the Clifford algebra Cl_3, i is basically duality operator which maps a vector to a bivector. This also explains the relation that product of three Pauli matrices give i which is algebraically same as e_{123} but geometrically is a volume element. It is a special element among the generators because it commutes with all other elements of the algebra, and hence along with the unit element belongs to center of the algebra. This element is also important because it is used in Hodge duality.

In the Clifford algebra Cl_3, i is basically the duality operator which maps a vector to a bivector.

4.3 *Proof of Pauli Matrix Identities*

In this section, we will prove identities of Pauli matrices without using the matrix algebra of Pauli matrices, but using Clifford algebra of Pauli matrices.

$$\sigma_j \sigma_k = \delta_{jk} + i \sum_l \epsilon_{jkl} \sigma_l \,. \tag{41}$$

To prove the above identity we will use Clifford multiplication.

$$\sigma_j \sigma_k = \sigma_j . \sigma_k + \sigma_j \wedge \sigma_k \,. \tag{42}$$

Now, we will use the mapping between the exterior product and the cross product using the Hodge duality.

$$\sigma_j \wedge \sigma_k = \sigma_{123} (\sigma_j \times \sigma_k) \,, \tag{43}$$

where we used the duality operator, also called pseudoscalar σ_{123} Using the above equation we get,

$$\sigma_j \sigma_k = \sigma_j . \sigma_k + \sigma_{123} (\sigma_j \times \sigma_k) \,. \tag{44}$$

Using the definition of cross product we get,

$$\sigma_j \sigma_k = \sigma_j . \sigma_k + \sigma_{123} \sum_l \epsilon_{jkl} \sigma_l \,. \tag{45}$$

Now, the volume element squares to -1 like i, so we can replace it. But the important thing over here is that geometrically σ_{123} is not the same as i. Hence we get the identity of Pauli matrices,

$$\sigma_j \sigma_k = \sigma_j . \sigma_k + i \sum_l \epsilon_{jkl} \sigma_l \,. \tag{46}$$

Now, we will prove another famous identity of Pauli matrices using Clifford algebra of Pauli matrices.

$$(\sigma . A) (\sigma . B) = A . B + i \sigma . (A \times B) \,. \tag{47}$$

To prove this identity, we will use Clifford product of Pauli matrices.

$$(\sum_j \sigma_j A_j)\,(\sum_k \sigma_k B_k) \tag{48}$$

$$= \sum_{jk} (\sigma_j \sigma_k)\,(A_j B_k) \tag{49}$$

$$= \sum_{jk} (\delta_{jk} + i \sum_l \epsilon_{jkl} \sigma_l)\,(A_j B_k) \tag{50}$$

$$= \sum_{jk} \delta_{jk} A_j B_k + \sum_{jkl} i \epsilon_{jkl} \sigma_k A_j B_k \tag{51}$$

$$= A.B + i\sigma.(A \times B)\,. \tag{52}$$

5. Conclusion

In this article we have shown that Clifford algebra is the algebra of spin and Pauli matrices form a representation of this algebra. All the properties of Pauli matrices follow from the underlying algebra. Though Pauli matrices give a faithful representation of Clifford algebra, many things miss out in this formalism. One is the geometrical nature of Clifford algebra. In Pauli formalism, there is no way to understand the algebraic properties of Pauli matrices, rather one takes them as axioms. While in Clifford algebra, all these algebraic properties follow from the geometry. Similarly, graded structure of Clifford algebra is not manifested in Pauli matrix formalism. Since spin is a fundamental property of matter and fields, we need to treat it in its natural algebraic formalism so that we have a better understanding. We anticipate that Clifford algebraic approach to spin can have lots of applications in diverese fields where spin plays a prominent role – from spintronics, spin networks and quantum spin systems to quantum computation. After the discovery of topological insulators, Clifford algebra has gained a lot of interest because it is central to the classification of topological phases of matter [14].

Acknowledgement

I would like to thank Louis H Kauffman for the beautiful discussions which have deepened my understanding of Clifford algebra. I also thank N S Vidhyadhiraja who gave me the opportunity to present this work to him, and for his kind and insightful remarks and critical comments. I also thank Joseph Samuel for the discussions which are related to this work. I would like to thank JNCASR for the various facilities which were used for this work.

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Address for Correspondence
Rukhsan-Ul-Haq
Room No. 316, Students Residence
Jawaharlal Nehru Centre for Advanced Scientific Research
Jakkur
Bengaluru 560 064, India.
Email: mrhaq@jncasr.ac.in