Hermann Weyl and Representation Theory

B Sury

He invented the character formula, the unitarian trick,
and decoded uniform distribution – pretty quick;
told us to hear the volume of a drum
but, about the shape, he said it was mum.
His favourite contribution? Take your pick!

A mathematcian with a physicist's guile,
and a global viewpoint – his signature style;
be it number theory, eigenvalues, Lie groups,
by his grace, readily jumped through hoops;
such was the magic of Hermann Weyl!

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Weyl was a universal mathematician whose fundamental contributions to mathematics encompassed all areas, and provided a unification seldom seen. His work on the theory of Lie groups was motivated by his life-long interest in quantum mechanics and relativity. When Weyl entered Lie theory, it mostly focussed on the infinitesimal, and he strove to bring in a global perspective. Time and again, Weyl's ideas arising in one context have been adapted and applied to wholly new contexts. In 1925–26, Weyl wrote four epochal papers in representation theory of Lie groups which solved fundamental problems, and also gave birth to the subject of harmonic analysis of semisimple Lie groups. In these papers, Weyl proved complete reducibility theorems and introduced many techniques which have become the standard way to study representations of Lie groups and their various generalizations in the last seven decades. Weyl's work covers several parts of mathematics, as well as parts of physics. In this article, we discuss mainly his contributions to the representation theory of Lie groups via the four papers mentioned above.

Keywords
Lie groups, gauge theory, epigenvalues, unitarian trick, character formula, Peter–Weyl theorem.

1. Weyl's Philosophy of Mathematics.

"The gods have imposed upon my writing the yoke of a foreign language that was not sung at my cradle" – these famous words appear in the preface to the celebrated book on classical groups by Hermann Weyl. Weyl was a poet, a visionary scientist, and a universal mathematician – all bundled into one. His contributions to mathematics encompassed all areas, and provided a unification seldom seen in the work of mathematicians. Weyl strove to bring in a global perspective to the theory of Lie groups which, at that time, focussed on the infinitesimal. The spectral properties of differential operators like the Laplacian – especially their asymptotics, was a refrain that kept surfacing in his work. This aspect has grown in riches with Mark Kac and others who discovered far-reaching connections between isopectrality and isometricity. After his time, these ideas have led to the so-called elliptic operators proof of the Atiyah–Patodi–Singer theorem. An asymptotic result on the count of eigenvalues proved by him for a bounded domain in 3D-space is known as Weyl's law and has been generalized to diverse contexts like automorphic forms. For instance, such an asymptotic formula proved for Riemann surfaces by Atle Selberg is of fundamental importance in showing the existence of abundantly many Maass forms for congruence subgroups in $SL(2, \mathbf{R})$.

His enduring interest in general relativity prompted him to propose a method to explain electromagnetism and Maxwell's equations via Einstein's theory. This failed as the physics did not support it – as Einstein said, if Weyl was right, the size of a particle would depend on its past history which would contradict experiments showing that all hydrogen atoms had identical properties. However, Weyl's ideas were reinterpreted by Kaluza using quantum mechanics, and this led to gauge theory[1], a flourishing subject today. Apart from their importance in particle physics, these non-abelian gauge theories have led to important mathematical work on four manifolds by Simon Donaldson and others.

[1] Joseph Samuel, Of Connections and Fields - I, *Resonance*, Vol.10, No.4, pp.10–21, 2005.

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As mentioned above, Weyl was the first to take a giant step in relating the spectral properties of the Laplacian of Riemannian manifolds[2] with their geometry and topology. In 1911, he considered a conjecture due to Lorentz and Sommerfeld, and showed that the dimension and volume of a bounded domain M in the Euclidean space of dimension 2 or 3 is determined by its Dirichlet or Neumann spectrum. More precisely, Weyl proved that the number of eigenvalues not exceeding λ is asymptotically of the form $\frac{Vol(B_n)Vol(M)}{(2\pi)^n}\lambda^{n/2}$, where B_n is the unit ball in Euclidean n-space. Nowadays, this is called Weyl's Law. The law has now been generalized to closed Riemannian manifolds, and apart from the spectrum determining the volume and dimension, it also determines the total scalar curvature.

[2]Harish Seshadri and Kaushal Verma, The Embedding Theorems of Whitney and Nash, *Resonance*, Vol.21, No.9, pp.815–826, 2016.

In 1925–26, Weyl wrote four path-breaking papers in representation theory which apart from solving fundamental problems, also gave birth to the subject of harmonic analysis of semisimple Lie groups. In these papers, Weyl proved complete reducibility theorems and introduced many techniques like 'character formulae' which have become the standard way to study representations of Lie groups and their generalizations. In 1926, along with F Peter, Weyl proved what he had announced in a 1925 paper, now known as the Peter–Weyl theorem. Weyl's abiding interest in differential equations played a role in the proof of the Peter–Weyl theorem. There is a masterful exposition on Weyl's work on Lie theory by A Borel, and a volume on Weyl's legacy edited by Katrin Trent. The reader is referred to these for detailed and insightful descriptions of Weyl's work in diverse areas; for representation theory, I have followed the excellent essays by Thomas Hawkins mentioned at the end of this article.

Weyl, as we mentioned, was a universalist who strongly believed in the unity of mathematical methodology. In this, he was a true disciple of Hilbert[3]. Both believed in, and followed what is sometimes known as the Riemannian doctrine whose essence is that proofs should be driven solely by ideas and not by calculations. This was exemplified by Weyl's work time and again; for instance, ideas from Riemann surfaces and covering spaces were brought to bear on the algebraic problem of complete re-

[3]Rajendra Bhatia, David Hilbert, *Resonance*, Vol.4, No.8, pp.3–5, 1999.

Weyl believed in the doctrine that proofs should be driven solely by ideas and not by calculations.

ducibility. In his book on group theory and quantum mechanics, Weyl considered for the first time, quantum systems in which the configuration spaces are replaced by abelian groups in duality – including finite abelian groups! Later, Andre Weil[4] took over the study of these Weyl systems over locally compact abelian groups in duality in a path-breaking work. This led to the study of the so-called Weil (or oscillator) representation, and choice of the 'adele group' as the abelian group enabled reinterpretation of Siegel's work on quadratic forms over number fields.

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[4] Kapil H Paranjape, Andre Weil, *Resonance*, Vol.4, No.5, pp.3–5, 1999.

Weyl developed general ideas leading to the synthesis of parts of mathematics, and had a gift for bringing to focus notions and naming them! He was the first to define vector spaces; the first to use the adjective 'symplectic' for the groups now known under that name; the first to coin the phrase 'maximal torus', and the first to use the phrase 'Lie algebra'. He had a flair for analogy. At one point, when defining group action on characters by $(g\chi)(x) = \chi(g^{-1}x)$, he likened the process to shoes and socks – the latter are worn first but the former are removed first!

2. Weyl's Unitarian Trick

The top practitioners of Lie theory at that time usually dealt with infinitesimal aspects. They made little distinction between a (global) Lie group and its Lie algebra. Weyl, being a globalist, was the first to break new ground, not only exposing the subtleties while making a distinction, but also pointing out the way to accomplish the desired goals globally. Weyl's unitarian principle is essentially an outcome of the principle that a holomorphic function which is zero on the real line must be the zero function.

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Consider the 'special linear' group $SL(n, \mathbf{C})$ of $n \times n$ complex matrices with unit determinant. Weyl was interested in describing the representation theory of this group completely. The group $SU(n)$ of $n \times n$ unitary matrices of unit determinant is a subgroup of the special linear group. This is a compact group which makes it possible to adapt methods from finite group theory by replacing addition by integration with respect to a translation-invariant measure/volume. On the other hand, infinitesimally, it is (that is, the tangent space of $SU(n)$ at the identity is) the space $su(n)$ of $n \times n$ skew–Hermitian matrices of trace zero. The infinitesimal version of $SL(n, \mathbf{C})$ is the space $sl(n, \mathbf{C})$ of all $n \times n$ complex matrices of trace zero (the determinant 1 condition becomes trace 0 condition). These tangent spaces at the identity also have a 'bracket' operation which is skew-commutative and give a structure of a so-called Lie algebra.

Unitarian trick in nascent form

If every finite-dimensional representation of $su(n)$ is completely reducible (that is, is a direct sum of irreducible representations), then the same property holds for finite-dimensional representations of $sl(n, \mathbf{C})$.

The proof of this is exceedingly simple. Given any finite-dimensional representation space V of $sl(n, \mathbf{C})$, and a sub-representation space W, by complete reducibility for $su(n)$, there is a complementary subspace W' of W in V which is a representation of $su(n)$. Thus, taking a vector space basis of V as the union of bases for W and W' respectively, the matrices of the representation with respect to such a basis are block matrices of the form,

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where A is a square matrix of size dimension of W. Moreover, for each element matrix $M \in sl(n, \mathbf{C})$, each entry of $B(M)$ is a linear homogeneous expression $\sum_{i,j=1}^{n} a_{ij} m_{ij}$. As W' is transformed to itself, the B-components of matrices coming from $su(n)$ are the zero block; that is,

$$\sum_{i,j} a_{ij} m_{ij} = 0 \ \forall \ m \in su(n).$$

Using the fact that $\overline{m_{ij}} = -m_{ji}$ for all i, j, it is easily deduced that $a_{ij} = 0$ for all i, j. Hence W' is also a $SL(n, \mathbf{C})$ sub-representation of V.

The unitarian trick for su(n) was already obtained by Schur a year before Weyl's 1925 paper, but Weyl went much further.

2.1 *Local to Global*

The unitarian trick stated above for $su(n)$ was already obtained by Schur a year before Weyl's 1925 paper, but Weyl went much further as we will show presently. The unitarian trick as stated above addresses the infinitesimal version and required only linear algebra (a subject that Weyl brought into mainstream mathematics by repeated applications). In order to do this over the group, one is faced with a task involving additional problems to which Weyl first brought attention and then eventually solved. The main question is whether all (Lie algebra) representations of $su(n)$ are 'integrable' – that is, whether they come by differentiating representations of the group $SU(n)$ at the identity. Using the exponential map of matrices, Weyl showed that a Lie algebra representation can be integrated to a representation defined on a small neighbourhood of the identity in the group $SU(n)$. However, when one tries to extend it to the whole group, one encounters problems and the natural extensions might be many-valued. Armed with his understanding of covering spaces through uniformization theory for Riemann surfaces, Weyl noted that one can define a single-valued extension on a covering space. Thus, if the covering spaces were compact there was no problem. Weyl actually showed that $SU(n)$ has no proper covering[5]. In this manner, Weyl could deduce that,

[5] Means it is simply-connected

Every finite-dimensional representation of $SL(n, \mathbf{C})$ is completely reducible.

As he mentioned, the real problem lies in the topological realm, and a flawed application of the 'integration' method would lead to false statements about complete reducibility. For instance, the universal cover of the compact circle group is the non-compact group of real numbers which has finite a dimensional representations like,

Armed with his understanding of covering spaces through uniformization theory for Riemann surfaces, Weyl noted that one can define a single-valued extension on a covering space.

$$t \mapsto \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

which is not completely reducible.

Weyl next generalized the unitarian trick and the complete reducibility theorem to include all the classical groups. Incidentally, the nomenclature 'classical groups' has never been precisely defined but Weyl was the first one to use it in his book with that title, which dealt with the special linear, symplectic, orthogonal and unitary groups. Weyl had these results by October 1924. After Weyl communicated the above results to Issai Schur, the latter responded on 10th November 1924 in a long 14-page letter wherein Schur mostly talked about his own work on characters. Schur mentions that it would be of considerable interest to see how his recent results on the characters of the rotation group could be derived from E Cartan's methods. He does compliment Weyl for his 'very beautiful' results at the end of the letter, but does not elaborate more. Weyl seems to have been inspired by this challenge, and went about creating the edifice of character theory for general semisimple groups with great energy. So much so that he was able to communicate to Schur that he had solved the representation problem in generality to which Schur replied on 26th November. In his official letter of 28th November, Weyl announces to Schur that, "By a modification of your beautiful method I have succeeded in solving the representation problem, including the calculation of the characters, for all simple and semisimple groups." Schur's reply of 26th November (to Weyl's informal communication to him some time between the 10th and the 26th) reads, "I am filled with the highest admiration for what you have now achieved – and in such a brief time. Your results are among the most beautiful and most important that have been attained in the field of group theory."

2.2 *Compact Real Form*

The unitarian trick and complete reducibility for general complex semisimple Lie groups G was accomplished by Weyl using two ideas, the first of which is the existence of a 'compact real form' of the Lie algebra. In other words, the Lie algebra $\mathfrak{g}$ of a complex semisimple group admits bases for which the structure constants are real. Any such basis would give a real form of $\mathfrak{g}$. For the unitarian trick, one needs a compact real form. Cartan had already noted in 1914 that the Killing form of a complex semisimple Lie

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algebra is negative-definite. Weyl seems to have been unaware of Cartan's paper and proved this again by himself. While doing this, he introduced for the first time the Killing form, explicitly as a trace function. Using the adjoint representation to identify the real form, and using the fact that this leaves the Killing form (which is definite) invariant, one has a compact real form. The second part of the complete reducibility theorem for semisimple groups is the proof by Weyl that the universal covering of a compact semisimple group is also compact – in other words, it is a finite cover. To deduce this, Weyl proved that maximal tori are all conjugate; a fact he said "coincides with known algebraic facts in case of classical groups."

2.3 *Weyl Groups Enter*

Weyl introduced the Killing form for the first time as a trace function.

Cartan's classification of simple Lie algebras depended on the properties of the so-called Cartan integers. Weyl reinterpreted the Cartan integers in such a manner that many properties of Cartan integers obtained earlier by Killing and Cartan by complicated calculations were easily deduced in a conceptual and transparent way. For a so-called Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$, the Killing form provides a duality between $\mathfrak{h}$ and its dual. Weyl observed that the effect of a linear form ξ on $\mathfrak{h}$ applied to a basis vector H_α of $\mathfrak{h}$ corresponding to a root α of the Lie algebra $\mathfrak{g}$ can be realized as the bilinear product $<\xi, \alpha>$, with respect to the dual of Killing form restricted to $\mathfrak{h}$. Weyl introduced the finite group of orthogonal transformations generated by 'reflections',

$$\sigma_\alpha : \xi \mapsto \xi - \frac{2 < \xi, \alpha >}{< \alpha, \alpha >}\alpha.$$

Weyl proved the existence of a basis of the Lie algebra with special properties which enabled him to explicitly exhibit a compact real form.

The finite group generated by these reflections of order 2 is now known as the Weyl group of $\mathfrak{g}$. As Weyl demonstrated, the determination of semisimple groups is based on that of the associated finite (Weyl) groups. With the aid of his interpretation of Cartan integers in terms of the above (dual of) Killing form restricted to $\mathfrak{h}$, Weyl could show that the basis of the Lie algebra $\mathfrak{g}$ could be chosen in a special form. A basis could be so chosen that the structure constants occurring in the commutator of basis elements have remarkable properties. This explicitly exhibits a compact real form of $\mathfrak{g}$, which is then used to carry out the unitarian trick for general semisimple $\mathfrak{g}$. The group version of the complete reducibility theorem was proved by Weyl by exploiting the role of maximal tori. He used properties such as the surjectivity of the exponential map on tori and the conjugacy theorem that every element of a compact semisimple Lie group lies in a conjugate of any fixed maximal torus. As mentioned earlier, the last-mentioned property enabled Weyl to prove that the universal cover of a compact semisimple Lie group is finite-sheeted[6]. Incidentally, finiteness of the dimension of invariants under polynomial representations is nowadays derived from complete reducibility, but Weyl does not seem to have realized this at that time.

[6]Means the fundamental group is finite.

3. Weyl Character Formula

3.1 *Elie Cartan's Approach*

Through his character theory, Weyl accomplished an integration of Cartan–Killing's 'theory of weights' approach with the Frobenius–Schur theory of characters. E Cartan had showed that an irreducible representation (V, π) of a complex semisimple Lie algebra $\mathfrak{g}$ naturally decomposes the space V into common eigenspaces under a Cartan subalgebra $\mathfrak{h}$ as the latter is commutative. The non-zero eigenspaces are called 'weight spaces' and the corresponding linear forms on $\mathfrak{h}$ are called 'weights' of the representation. The weights of the adjoint (= conjugation) action are called 'roots', and the weight spaces are called 'root spaces' of the Lie algebra. For an irreducible representation π, the 'weights' are ordered 'lexicographically'; that is, fixing a basis $\{H_1, \cdots, H_r\}$ of $\mathfrak{h}$, then $\mu > \lambda$ provided the smallest $k \leq r$ such that $(\mu - \lambda)(H_k) \neq 0$ is positive. The largest in this order is called the 'highest weight' of the representation π. Cartan also proved that the set $Q(\pi)$ of weights of π is conjugation-invariant and all conjugates have the same multiplicity. Weyl noted that this can be interpreted in terms of the finite (Weyl) group W as $Q(\pi)$ being $W-$stable and λ and $w(\lambda)$ having the same multiplicity for all $\lambda \in Q(\pi)$ and $w \in W$. He introduced the set of all integral combinations $\mu := \sum_{k=1}^{r} m_k \lambda$ of the basic weights $\lambda_1, \cdots, \lambda_r$ of π which have the property that $2 < \mu, \alpha > / < \alpha, \alpha >$ is an integer for each root α; any such linear combination occurs as a weight in an irreducible representation of $\mathfrak{g}$. For this reason, one calls the totality of all such integral combinations with the integrality property as the set Λ of all weights. It is a free abelian group with a basis having $dim(\mathfrak{h})$ elements. Cartan's work tells us that there are certain fundamental weights such that an element of Λ is a highest weight of an irreducible representation if, and only if, it is a non-negative linear combination of these fundamental weights. This subset Λ_{dom} is called the set of dominant weights because $\lambda \geq w(\lambda)$ for all $\lambda \in \Lambda_{dom}, w \in W$.

3.2 *Maximal Tori to the Fore*

Frobenius and Schur developed the character theory of compact, simply-connected Lie groups and obtained orthogonality relations. Weyl focussed on the universal cover $\widetilde{G_u}$ of the compact real form G_u of G he had defined. Carrying over his idea behind the unitarian trick which shows that the irreducible representations of the Lie algebra $\mathfrak{a}_u$ of G_u are restrictions of irreducible representations of the Lie algebra of G, Weyl observed that the irreducible characters of $\widetilde{G_u}$ are the restrictions of irreducible characters of the universal cover of the adjoint group of G. Consequently, Frobenius–Schur orthogonality relations can be expressed as:

$$\int_{\tilde{G}_u} \chi_\lambda(g)\overline{\chi_\mu(g)}dg = 0 \ \ or \ \ Vol(\widetilde{G_u}),$$

according to whether the irreducible characters λ, μ are different or the same, where $\mathrm{Vol}(\widetilde{G_u})$ is the volume of $\widetilde{G_u}$ under a translation-invariant measure on it. Schur's orthogonality relations were further transformed by Weyl into a form more amenable to applications. He realized that the conjugacy theorem was valid for the universal cover $\widetilde{G_u}$ as well. Therefore, every element of $\widetilde{G_u}$ has a conjugate in a maximal torus T; as characters do not change under conjugacy, it suffices to evaluate them on T. As the

Weyl transformed Schur's orthogonality to a much more amenable form.

exponential map is an isomorphism onto the torus, the irreducible characters χ_λ of a representation Π_λ could be expressed on $t \in T$ as:

$$\chi_\lambda(t) = tr\,\Pi_\lambda(t) = tre^{\pi_\lambda(X)},$$

where $t = \exp(X)$ and π_λ is the derivative representation of Π_λ. For a vector v in the μ-weight space (that is, $\pi_\lambda(H)(v) = \mu(H)v$ for all H), we have $\exp(\pi_\lambda(X))(v) = e^{\mu(X)}v$; this means that $\Pi_\lambda(t)$ is a diagonal matrix with trace,

$$\chi_\lambda(t) = \sum_{\mu \in \Pi(\lambda)} m(\mu) e^{\mu(X)},$$

where $m(\mu)$ denotes the multiplicity of μ and the sum is over the weights of the representation with highest weight λ. Note that there may be more than one X with $\exp(X) = t$ but the value $e(\mu) := \exp(\mu(X))$ is the same for all of them.

3.3 *Finite Fourier Series*

At this point, once again Weyl's genius for recognizing the essential unity of mathematical methodology comes to the fore. In the above notation, $e(\mu) = \exp(2i\pi\phi_\mu(X))$, where $\phi_\mu(X) = \mu(\sum_k \phi_k X_k$ and $X_k = adH_k$ so that Weyl was able to express the character on T as a finite Fourier series[7] with integer coefficients. As $e(\mu)e(\nu) = e(\mu+\nu)$, these finite Fourier series can be added and multiplied and the Weyl group acts on these series naturally. These sums can be regarded as elements of the so-called integral group ring $\mathbf{Z}[\Lambda]$ consisting of formal integer combinations of the weights. Viewing the Fourier series of the irreducible characters restricted to T,

[7] S Thangavelu, Fourier Series: The Mathematics of Periodic Phenomena, *Resonance*, Vol.1, No.10, pp.44–55, 1996.

Weyl re-expressed the orthogonality relations in terms of the so-called rotation angles associated to any element of T (these are $\phi_1, \cdots, \phi_r \in [0,1]$ where $t = \exp(adH)$ and $H = 2i\pi \sum_{k=1}^r \phi_k H_k$ in the Lie algebra of G_u. If one writes t as $t(\phi_1, \cdots, \phi_r)$, then $t(\phi_1, \cdots, \phi_r) = t(\psi_1, \cdots, \psi_r)$ whenever $\phi_k - \psi_k \in \mathbf{Z}$ (We fixed the domain for ϕ_i's to be [0,1] so that the expression for t is well-defined.). By the conjugacy theorem, every element of $\widetilde{G_u}$ (being conjugate to an element of T) itself has a set of rotation angles

Weyl's genius for synthesis of different themes is illustrated by the way he brought in finite Fourier series to the study of representations of compact Lie groups.

$\phi_1, \cdots, \phi_r$ in [0, 1] and Weyl expressed the volume element of $\widetilde{G_u}$ as a product over the roots of the Lie algebra as:

$$\prod_{\alpha \in \Phi} [e(\alpha) - 1] (d\phi_1) \cdots (d\phi_r),$$

because $e(\alpha) = \exp(2i\pi\alpha(\phi_1 H_1 + \cdots + \phi_k H_k))$ is a function of $\phi_1, \cdots, \phi_r$.

3.4 *Denominator Formula and Antisymmetric Fourier Series*

Schur is supposed to have referred to the last-mentioned result of Weyl as 'exceedingly beautiful'. Note that,

$$\prod_{\alpha \in \Phi} [e(\alpha) - 1] = \Delta\overline{\Delta},$$

where $\Delta = \prod_{\alpha > 0}[e(\alpha/2) - e(-\alpha/2)]$, a product over positive roots, is called 'Weyl's denominator function'. The orthogonality of irreducible characters can be expressed thus as:

$$\int_0^1 \int_0^1 \cdots \int_0^1 (\chi_\lambda \Delta) \overline{(\chi_\mu \Delta)} (d\phi_1) \cdots (d\phi_r) = 0 \ \ or \ \ Vol(\widetilde{G_u}),$$

according to whether $\lambda \neq \mu$ or not. In other words, one would have an understanding of irreducible characters if one has an understanding of the product expressions $\xi_\lambda := \chi_\lambda \Delta$ for irreducible characters λ. Note that each 'simple reflection' w_{α_i} in W takes α_i to $-\alpha_i$, and leaves the set of other positive roots stable, changing Δ to $-\Delta$. Hence, the Weyl group permutations act on Δ by the sign of the permutation. In other words, both Δ and ξ_λ are alternating finite Fourier series with the latter having Δ as a factor. Hence, we have $\chi_\lambda = \xi_\lambda / \Delta$.

It is amazing that to understand certain symmetric functions which appear, Weyl expressed them in terms of two antisymmetric functions.

It is amazing that to understand the symmetric function χ_λ, it is easier to understand the antisymmetric function ξ_λ! To construct the ξ_λ, Weyl considered the antisymmetric sums corresponding to dominant weights μ defined as:

$$\omega_\mu = \sum_{w \in W} sgn(w)\, e(w\mu),$$

where $sgn\,(w)$ denotes the sign of the permutation $w \in W$. Weyl proved that Δ itself can be recognized in terms of these sums as:

$$\Delta = \omega_\rho \;\; where \;\; \rho = \frac{1}{2}\sum_{\alpha>0}\alpha.$$

Also, $\xi_\lambda = \omega_{\lambda+\rho}$ which gives the remarkably beautiful – 'Weyl's Character Formula',

$$\chi_\lambda = \frac{\omega_{\lambda+\rho}}{\omega_\rho} = \frac{\sum_{w\in W} sgn\,(w) e\,(w(\lambda+\rho))}{\sum_{w\in W} sgn\,(w) e\,(w(\rho))}\;.$$

This is to be thought of as an element of the integral group ring $\mathbf{Z}[\Lambda]$ by observing that the denominator is non-zero and divides the numerator in this group ring. In this manner, Weyl's character formula brings together Frobenius–Schur character theory and Cartan's theory of weights.

Neither Weyl nor Cartan knew of Schreier's work which proved the existence of a universal cover for general, locally Euclidean topological groups.

4. Peter–Weyl Theorem

4.1 *Pre-(Peter–Weyl)*

Ironically, the 'Weyl chambers' first appeared not in Weyl's papers but in Cartan's papers! Cartan refined Weyl's proof of finiteness of the fundamental group of a compact, semisimple group by using more of the geometry of the Euclidean space which occurs as the Lie algebra of a maximal torus, and he used Weyl chambers in this context. Tragically, neither Weyl nor Cartan knew of Schreier's work which proved the existence of a universal cover for general, locally Euclidean topological groups. Cartan became aware of Schreier's work after the latter's untimely death at the age of 27 due to general sepsis. Weyl was highly inspired and influenced by Cartan's extensive work on the theory of semisimple groups. In a letter to Cartan, written in March 1925, he says, "Since I became acquainted with the general theory of relativity, nothing has so stirred me and filled me with inspiration as the study of your works on continuous groups. I do not at all value

Weyl wrote to Cartan in March 1925 that, "Since I became acquainted with the general theory of relativity, nothing has so stirred me and filled me with inspiration as the study of your works on continuous groups. I do not at all value my present share in this theory very highly. I really seem to be only the fortuitous meeting point for you and Schur."

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4.2 *Analogue of Group Algebra*

A year after writing the above letter, Weyl published with F Peter[8] a sequel to his 1925 papers, which broke new ground in a direction quite different from Cartan or Schur. This is the famous Peter–Weyl theorem wherein they used the theory of integral equations. Being the universalist that he was, Weyl envisaged an analogue of the group algebra of finite groups which could play a similar important role for continuous groups. On a compact, simply connected real Lie group G, Weyl considered the space $C(G)$ of all complex-valued continuous functions, where the regular representation of the group could be represented. Looking at irreducible characters as class functions[9] which are orthonormal, Weyl knew that if one could prove their 'completeness' (meaning that each class function f satisfies the Parseval identity $\int_G |f(t)|^2 dt = \sum_n c_n^2$, where c_n's are the coefficients of f in the expansion of f in terms of the orthonormal set of irreducible characters), then Cartan's theorem on dominant weights giving finite-dimensional representations would follow. Having considered the above regular representation on $C(G)$, Weyl stated that decomposing the regular representation would lead to the result that the irreducible characters form a 'complete' orthonormal system.

[8] Was like a student, although he had already received a doctoral degree in physics working on indices of refraction and absorption constants of diamonds.

[9] Functions constant on conjugacy classes.

4.3 *Hilbert–Schmidt Theory*

In the paper with F Peter, Weyl proved this 'decomposition' theorem for compact groups admitting translation-invariant measures (at that time, the existence of Haar measure was not known in general). The continuous counterpart of the multiplication operation on the group algebra of a finite group suggested to them what is now known as the 'convolution' of functions. Though this notion was known through various examples, it was this paper of Peter and Weyl which signalled the advent of this operation into mainstream analysis. Moreover, 'decomposition' of infinite-dimensional spaces (such as $C(G)$) under operators (such as those coming from right/left multiplication on $C(G)$) was almost unknown; the known theory was by Hilbert and Schmidt and dealt with integral operators. Schmidt was a student of Hilbert and proved in his doctoral dissertation, the so-called Hilbert–Schmidt theorem asserting that for any symmetric integral operator T and a continuous function f, the continuous function Tf could be expressed as a uniformly convergent series in terms of the orthogonal eigenfunctions of T. However, for Peter and Weyl, the operators $T_f\,(f \in C(G))$ defined by $T_f(g) = f * g$ (that is, $T_f(g)(x) = \int_G f(xy^{-1})\,g(y)\,dy$) are not symmetric. However, as Schmidt had done earlier, Peter and Weyl considered the Hermitian symmetric operator $T_f T_f^*$ where T_f^* is the conjugate transpose of T_f. Now,

$$T_f^*(g)(x) = \int_G \overline{f(yx^{-1})}\,g(y)\,dy$$

$$= \int_G \widetilde{f}(xy^{-1})g(y)dy\,,$$

where $\widetilde{f}(x) = \overline{f(x^{-1})}$. In other words, for any $f \in C(G)$, the operator T_h, where $h = f * \widetilde{f}$, is a symmetric operator to which Hilbert–Schmidt theory applies. The right multiplication operation $R_f(g) = g * f$ and the left multiplication operation $T_{f*\tilde{f}}$ above commute; thus, the eigenspaces under the latter are carried to themselves by the latter. This is the basic reason behind the decomposition of the regular representation. The Peter–Weyl theorem can be stated in many versions, one of which parallels what Frobenius's theorem gives for finite groups. According to this, all irreducible representations of G are finite-dimensional, and as a right regular representation of G, the space $C(G)$ decomposes as a direct sum of $V \otimes V^*$, where the direct sum runs over the irreducible representations V. In other words, each irreducible representation of G occurs as a sub-representation of $C(G)$ exactly 'd' times, where 'd' is its dimension.

4.4 *SU(2)*

Let us demonstrate how these various formulae look like for the group $SU(2)$ which consists of the complex matrices $\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}$ which satisfy $|a|^2 + |b|^2 = 1$. As a space, this is nothing else than the unit 3-sphere in 4D-space. The diagonalization (spectral) theorem shows that every element of $SU(2)$ is conjugate to a diagonal matrix $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ for $\theta \in [0, 2\pi]$. These diagonal matrices form a maximal torus. Any linear representation π of $SU(2)$ on a finite-dimensional vector space V is determined by its character, which is the conjugacy-class invariant function $g \mapsto tr\,(\pi(g))$. Therefore, π is determined by the traces $tr\,(\pi(g))$ as g varies in the diagonal torus. The natural action π_n of $SU(2)$ on the vector space V_n of homogeneous polynomials in two variables X, Y of degree n is given by:

$$\pi_n(S)\,(f(X, Y)) = f\,(tX + uY, vX + wY)\,,$$

where $S^{-1} = \begin{pmatrix} t & u \\ v & w \end{pmatrix}$ and $S \in SU(2)$[10]. With respect to the ordered basis,

$$\{X^n, X^{n-1}Y, \cdots, XY^{n-1}, Y^n\}\,,$$

of the $(n + 1)$-dimensional space V_n, the matrix representation is,

$$\pi_n\,(diag\,(e^{i\theta}, e^{-i\theta})) = diag(e^{-in\theta}, e^{-i(n-2)\theta}, \cdots, e^{in\theta}).$$

Therefore, the character of the representation (π_n, V_n) on the diagonal torus is the function,

$$diag\,(e^{i\theta}, e^{-i\theta}) \mapsto \sum_{k=0}^{n} e^{i(n-2k)\theta} = \frac{\sin\,((n+1)\,\theta)}{\sin(\theta)}.$$

The fact that these representations π_n of $SU\,(2)$ are irreducible, follows from the fact that the functions $\phi_n(\theta) = \frac{\sin\,((n+1)\theta)}{\sin\,(\theta)}$ (these are essentially Chebychev polynomials of the second kind) form an orthonormal set with respect to the inner product,

$$< f, g >= \frac{2}{\pi} \int_0^{\pi} f\,(\theta) g\,(\theta) \sin^2\,(\theta) d\theta\,,$$

[10] See the philosophy of 'shoes and socks' mentioned in the first section.

on the space of square-integrable functions on $[0, \pi]$ with respect to the above product. Peter–Weyl theorem for $SU(2)$ amounts to the statement that these are all the irreducible representations – that is, this family is 'complete'. This can be seen directly as follows. Let ϕ be any square-integrable function on $[0, \pi]$ with respect to the above inner product. Consider the 'odd' function on $[-\pi, \pi]$ defined by:

$$\chi(\theta) = \phi(\theta) \sin(\theta) \;\; for \;\; \theta \in [0, \pi],$$

$$\chi(\theta) = \phi(-\theta) \sin(\theta) \;\; for \;\; \theta \in [-\pi, 0].$$

Look at the Fourier series expansion,

$\chi(\theta) = \sum_r c_r e^{ir\theta}$; then $c_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi(\theta) e^{-ir\theta}$.

As χ is an odd function, we get $c_{-r} = -c_r$ so that we obtain $c_0 = 0$, and for $\theta \in [0, \pi]$, we obtain,

$$\chi(\theta) = \phi(\theta) \sin(\theta) = \sum_{r \geq 1} c_r 2i \sin(r\theta) = \sum_{n \geq 0} 2ic_{n+1} \sin((n+1)\theta).$$

Hence, $\phi(\theta) = \sum_{n \geq 0} 2ic_{n+1} \phi_n(\theta)$ for $\theta \in [0, \pi]$. Thus, the family $\{\phi_n\}_n$ is a complete orthonormal family.

5. Miscellanea

5.1 *Invariant Theory*

The principal motivation for Weyl to write his book on classical groups was to develop representation-theoretic methods to derive theorems of classical invariant theory. According to Weyl, Hilbert had solved the main problems (Hilbert's basis theorem), and had almost killed the whole subject (of invariant theory) "but its life still lingers on, however flickering during the next decades. In recent times, the tree of invariant theory has shown new life, and has begun to blossom again, chiefly as a consequence of interest in invariant-theoretic questions awakened by the revolutionary developments in mathematical physics (relativity theory and quantum mechanics), but also due to the connection of invariant

A cornerstone in Weyl's discussion is the so-called first fundamental theorem of invariant theory.

theory with the extension of the theory of representations to continuous groups and algebras."

A cornerstone in Weyl's discussion is the so-called first fundamental theorem of invariant theory. On a vector space V, one has the full general linear group $GL(V)$ of all invertible linear transformations. On the n-fold tensor product $V^{\otimes^n}$ of V with itself, apart from the action of $GL(V)$, there is also the natural action of the permutation group S_n on n symbols (which simply permutes the factors). The remarkable result proved by Schur is that the subalgebras of the full algebra of linear transformations of $V^{\otimes^n}$ generated by these two actions are mutual centralizers[11] Using Schur's result, Weyl showed how to decompose the tensor product under the action of the product $GL(V) \times S_n$. He also analyzed this question for other 'classical' groups.

[11] Means each set is the full set of operators commuting with the other set.

5.2 *Finite Rotation Groups*

Weyl came up with a beautiful argument to obtain a classification of the finite subgroups of $SO(3)$the 3×3 real orthogonal matrices with determinant 1. Each nontrivial matrix in $SO(3)$ fixes a line (an axis) around which it acts as rotations. Given a finite subgroup G of $SO(3)$, one considers the set S of points on the unit sphere which lie on an axis of rotation for some nontrivial element of G. For each axis of rotation of elements of G, the minimal angle of rotation of an element is of the form $2\pi/n$, and the subgroup of G with that axis is isomorphic to the cyclic group of order n. Such an axis is called an n-fold axis. If S breaks up into orbits $O_1, \cdots, O_r$ under G, there are positive integers $n_1, \cdots, n_r$ such that O_i consists of those points of S which lie on n_i-fold axes. Thus, the counting formula for group actions gives the equation:

$$\sum_{i=1}^{r} \left(1 - \frac{1}{n_i}\right) = 2 - \frac{2}{O(G)}.$$

This Diophantine equation implies $r = 2$ or 3 and can be solved completely to yield the full list of finite subgroups up to conjugacy:

(i) The symmetries of an n-fold axis – a cyclic group of order n.

(ii) The symmetries of an n-fold axis and n two-fold axes orthogonal to it – a dihedral group of order $2n$.

(iii) The symmetries of a tetrahedron – isomorphic to A_4.

(iv) The symmetries of a cube – isomorphic to S_4, and

(v) The symmetries of an icosahedron – isomorphic to A_5.

In conclusion, one cannot do better than quote Weyl himself who said (according to Raoul Bott), "The problems of mathematics are not problems in a vacuum. There pulses in them the life of ideas which realize themselves in concreto through our human endeavours in our historical existence, but forming an indissoluble whole transcending any particular science."

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Suggested Reading

[1] Thomas Hawkins, *Emergence of the theory of Lie groups, An Essay in the History of Mathematics 1869–1926*, Springer-Verlag, 2000.

[2] *Hermann Weyl 1885–1985*, Ed. K Chandrasekharan, Springer-Verlag, 1986.

[3] *Gesammelte Abhandlungen Hermann Weyl*, Ed. K Chandrasekharan, Springer-Verlag, 1968.

[4] *Groups and Analysis: The legacy of Hermann Weyl*, Ed. by Katrin Trent, Cambridge University Press, 2008.