

## The RC Circuit: An Approach with Fourier Transforms

Mitrajyoti Ghosh

83, Mitrapara 2nd Lane, Harinavi,  
Kolkata 700148, West Bengal,  
India.

Email:

mijospeakingnow@gmail.com

In this article we shall mathematically analyse the Resistor-Capacitor (RC) circuit with the help of Fourier transforms (FT). This very general technique gives us a lot of insight into solving first order differential equations with source terms depending on time. In itself, the RC circuit is by far the most commonplace entity in modern electronics. But the method of FT is not the accepted custom for an electronic engineer, who is probably more comfortable working with complex impedances and phasors while solving problems in network analysis. In fact, what is used much more extensively is the Laplace transform. But a lot of things, (including the complex impedance itself, and some insight into complex analysis) can be understood better if we use the FT approach to solve the differential equations that come up in network analysis. The use of FT comes smoothly from first principles – precisely what we set out to demonstrate here.

We start with the circuit shown in *Figure 1*, with the initial conditions that at  $t = 0$ , charge  $q$  across the capacitor is 0, and the current  $i = \frac{dq}{dt} = 0$ .

We shall also impose the very important condition that  $V_{in} = 0 \forall t < 0$

The importance of this condition shall be clear in due course as we look at the methods employed to solve this differential equation for different forms of the input voltage applied.

### 1. The RC Circuit and its Differential Equation

For the circuit shown in *Figure 1*, the differential equation for charge  $q$  on the capacitor is given by,

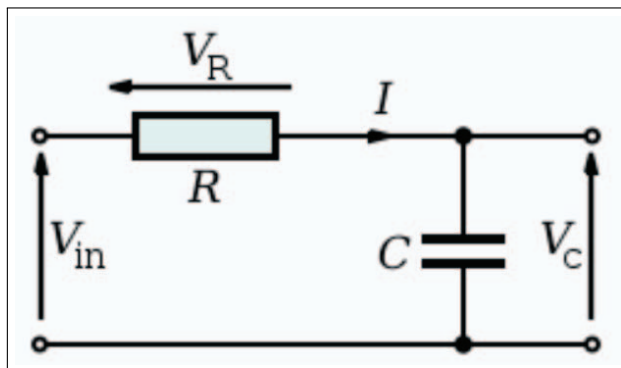
$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_{in}}{R}. \quad (1)$$

#### Keywords

Fourier transforms, contour integration, circuit theory.



**Figure 1.** The RC Circuit with time dependent input voltage  $V_{in}$ .



Note that this is a problem involving an **inhomogeneous differential equation with homogeneous boundary conditions**.

Let  $\tilde{q}$  be the FT of  $q$ .

Then we have,

$$q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{q} e^{i\omega t} d\omega, \tag{2}$$

$$\tilde{q} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q e^{-i\omega t} dt. \tag{3}$$

Thus,

$$\frac{dq}{dt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{q} i\omega e^{i\omega t} d\omega. \tag{4}$$

And also, we have the FT of  $V_{in}$  and the inverse transform,

$$\tilde{V}_{in} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V_{in} e^{-i\omega t} dt, \tag{5}$$

$$V_{in} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{V}_{in} e^{i\omega t} d\omega. \tag{6}$$

Solving  $q$  for different forms of  $V_{in}(t)$  can reveal many aspects of RC circuits and aid learning some subtleties of contour integration in physics along the way.

Using (2–6) and substituting them back into (1) we finally get,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \tilde{q} i\omega + \frac{\tilde{q}}{RC} - \frac{\tilde{V}_{in}}{R} \right) e^{i\omega t} d\omega = 0, \tag{7}$$

Hence,

$$\tilde{q} i\omega + \frac{\tilde{q}}{RC} - \frac{\tilde{V}_{in}}{R} = 0, \tag{8}$$



Where (8) is basically an analog of **Kirchhoff's Law in the frequency domain**, which gives us,

$$\tilde{q} = \frac{\tilde{V}_{in}C}{1 + i\omega RC}. \quad (9)$$

Therefore, by inverting the FT, the charge on the capacitor  $q$  can be written as,

$$q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{V}_{in}C}{1 + i\omega RC} e^{i\omega t} d\omega. \quad (10)$$

In the upcoming sections, we shall solve  $q$  for different forms of  $V_{in}(t)$  and thereby explore the RC circuit in greater detail. This will reveal certain aspects of RC circuits such as their functioning as differentiators, integrators and filters.

## 2. Some Simple Forms of $V_{in}$ to Illustrate Contour Integration

### 2.1 The Voltage Spike

Suppose there is a voltage spike at  $t = t_0$ .

Mathematically, we can model this using the Dirac delta function as,

$$V_{in} = A_0 \delta(t - t_0), \quad (11)$$

where  $A_0$  has suitable dimensions.

Replicating the strategy as shown in the previous section, we should obtain:

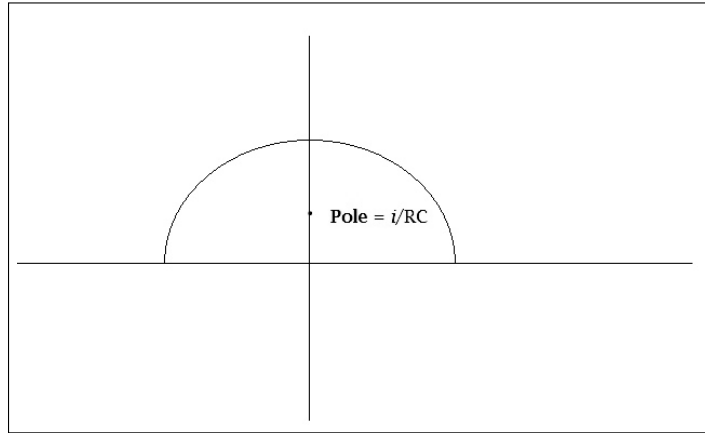
$$\tilde{V}_{in} = \frac{1}{\sqrt{2\pi}} A_0 e^{-i\omega t_0}, \quad (12)$$

$$q = \frac{A_0 C}{2\pi} \int \frac{e^{i\omega(t-t_0)} d\omega}{1 + i\omega RC} = \frac{A_0}{2\pi i R} \int \frac{e^{i\omega(t-t_0)} d\omega}{\omega - i/RC}. \quad (13)$$

This integral is clearly a complex integral and we need to move over to the complex  $\omega$ -plane to solve it. However,  $\omega$  is entirely real. Therefore we can best do by using the semi-circular contour (of radius  $R$  as shown below), for integration.



**Figure 2.** Closing the contour in the upper half plane for  $t > t_0$ .



**Jordan’s Lemma** for finding integrals of the form  $\int f(z)e^{ikz}dz$  says that for  $k > 0$ , the contour must be closed in the upper half plane (*Figure 2*) for the integration around the semicircular part of the contour to vanish as  $R \rightarrow \infty$ . This ensures that the only quantity we shall be left with, when  $R \rightarrow \infty$  is the integral along the real  $\omega$  line from  $-\infty$  to  $\infty$ .

For  $k < 0$ , the contour should be closed in the lower half plane for the integral on the semicircular contour to vanish.

In this case  $k \equiv (t - t_0)$

**For  $t > t_0$**  (In other words  $k > 0$ ):

There is only one pole  $\omega = \frac{i}{RC}$  enclosed by the contour. The residue at this pole is given by  $Res = e^{i(i/RC)(t-t_0)} = e^{-\frac{(t-t_0)}{RC}}$ .

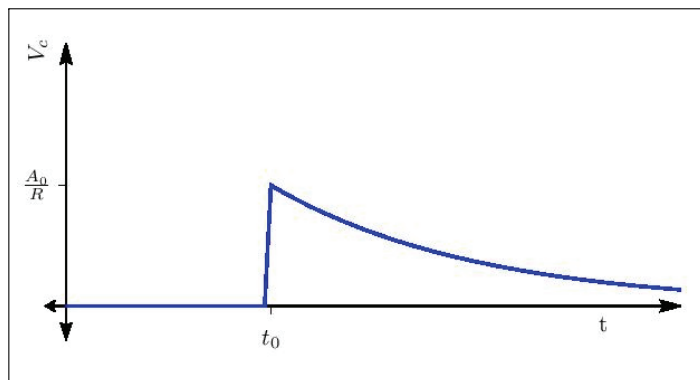
We obtain therefore, using the **residue theorem** of complex integration,

$$q = \frac{A_0}{2\pi i R} (2\pi i) e^{-\frac{(t-t_0)}{RC}} = \frac{A_0}{R} e^{-\frac{(t-t_0)}{RC}}. \quad (14)$$

**For  $t < t_0$ :**

The contour should be closed in the lower half plane. However there are no poles there. So the integral vanishes.





**Figure 3.** At  $t = t_0$ , the sudden spike delivers  $q_0 = A_0/R$  to the capacitor.

So finally we obtain,

$$\begin{aligned}
 q &= \frac{A_0}{R} e^{-\frac{(t-t_0)}{RC}} \quad (t > t_0) \\
 &= 0 \quad (t \leq t_0).
 \end{aligned}
 \tag{15}$$

This is surely the expected result, since at  $t = t_0$ , the huge voltage spike immediately charges the capacitor fully (*Figure 3*), and afterwards it just discharges exponentially, as it can be clearly seen from (15).

## 2.2 The DC Voltage Source

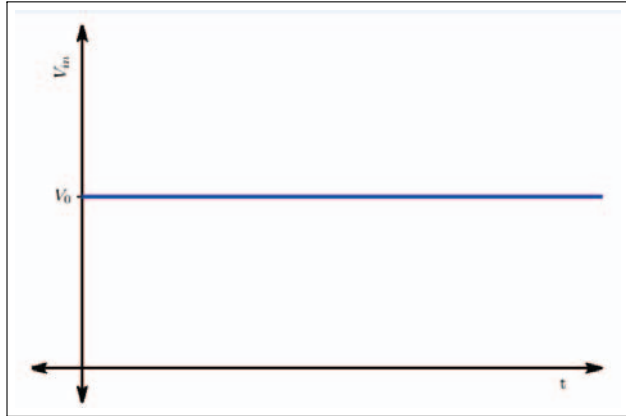
We consider next one of the most ubiquitous phenomena in electronics, a capacitor connected to a DC source. The form of  $V_{in}$  is shown below (*Figure 4*).

$$\begin{aligned}
 V_{in} &= V_0, \quad t > 0 \\
 &= 0, \quad t \leq 0.
 \end{aligned}
 \tag{16}$$

All it means is that at  $t = 0$ , the uncharged capacitor is connected to the DC source.



**Figure 4.** DC Step Voltage.



We shall again resort to the strategy of F.

$$\begin{aligned}\tilde{V}_{in} &= \frac{V_0}{i\omega\sqrt{2\pi}} \\ \therefore q &= -\frac{V_0C}{2\pi RC} \int \frac{e^{i\omega t} d\omega}{\omega(\omega - i/RC)}.\end{aligned}\quad (17)$$

The contour chosen for integration is the same but now, there are two poles, one at  $\omega = 0$  and the other at  $\omega = i/RC$ .

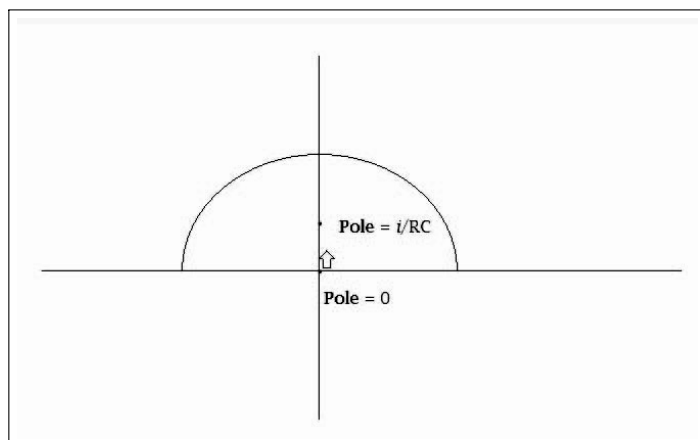
But now we have a problem. One of the poles (namely  $\omega = 0$ ) lies on the contour of integration itself. One way out of the problem is to distort the contour of integration slightly around the pole in a semicircular detour. Another way is to imagine that the pole is actually not on the contour but is slightly displaced along the imaginary axis by a tiny amount.

In the case of the latter, the contour integral would be written as,

$$q = \lim_{\epsilon \rightarrow 0} \left( -\frac{V_0}{2\pi R} \right) \int \frac{e^{i\omega t} d\omega}{(\omega - i\epsilon)(\omega - i/RC)}.\quad (18)$$

But the question is - do we shift the pole upwards or do we shift it downwards? In other words do we make  $\epsilon > 0$  (shifting upwards) or  $\epsilon < 0$  (downwards) ?





**Figure 5.** Two poles, one shifted upward to maintain causality.

(An analogous question if we use the detour around the contour would be - should we traverse the contour in such a way that the pole is within the boundary or outside?)

Let us say that we shift the pole downwards. Now, when for  $t < 0$ , the contour is closed downwards, there will exist a pole enclosed by the contour. If there is a pole, then we can compute a residue. If there is a residue, then it means that the integral is nonvanishing. But that is physically absurd, since for  $t < 0$ , there was no voltage in the circuit (this was explicitly stated earlier if you recall), and therefore it couldn't have had any charge circulating through it at all. Thus shifting the pole downward makes no causal sense – the circuit simply cannot respond before the input voltage is applied. To make the solution vanish  $\forall t < 0$ , all the poles must, if possible, be banished to the upper half plane (*Figure 5*).

The reader can see that, if the detour around the contour is employed to make sense of the integral, then for  $t > 0$ , the detour must be made in such a way that the pole is enclosed by the contour. For  $t < 0$ , the detour should be such that the contour excludes the pole.

Now we can solve the above integral using the residue theorem as before, by solving for  $t < 0$  and  $t > 0$ . For  $t > 0$ , the residues



obtained are  $-iRCe^{-t/RC}$  and  $iRC$  (here we have taken the limit  $\epsilon \rightarrow 0$ ), while for  $t < 0$ , the contour is closed in the lower half plane, where there are no poles and therefore no residues. Thus we get the expected,

$$\begin{aligned} q &= V_0C(1 - e^{-t/RC}), t > 0 \\ &= 0, t \leq 0. \end{aligned} \tag{19}$$

There is no response in the circuit before the capacitor is connected to the DC source. But once it is, the capacitor starts charging exponentially to a voltage  $V_0$ .

### 3. The Finite Wave train as Input

The finite wave train represents a very physical input voltage, one that is typically encountered in the lab. Consider the following form of  $V_{in}$ :

$$\begin{aligned} V_{in} &= V_0 \sin \omega_0 t, 0 \leq t \leq \frac{2N\pi}{\omega_0} (N \in \mathbb{Z}) \\ &= 0, \text{ elsewhere.} \end{aligned} \tag{20}$$

This is the most physical possible input, as no AC source works forever. There is also a certain time at which it is switched on, so before this instant, there is no emf in the circuit. This form of input is of interest also because it allows us some insight into the working of filter circuits that form the keystone of modern day telecommunication (*Figure 6*).

The strategy remaining the same, we find the FT,

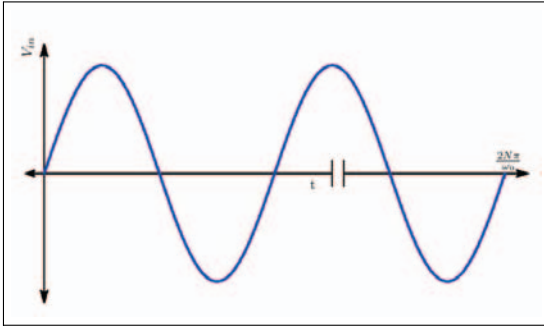
$$\tilde{V}_{in}(\omega) = \frac{V_0}{2\sqrt{2\pi}} \left[ \frac{1 - \exp\left(\frac{-2i\omega N\pi}{\omega_0}\right)}{\omega_0 - \omega} + \frac{1 - \exp\left(\frac{-2i\omega N\pi}{\omega_0}\right)}{\omega_0 + \omega} \right].$$

Upon inverting the FT to find  $q$ , we obtain the following,

$$q(t) = \frac{V_0}{4Ri\pi} (I_1 + I_2 + I_3 + I_4).$$







**Figure 6.** The finite wave train.

Where,

$$I_1 = \int \frac{d\omega e^{i\omega t}}{(\omega - i/RC)(\omega_0 - \omega)}. \quad (21)$$

$$I_2 = \int \frac{d\omega e^{i\omega(t - \frac{2N\pi}{\omega_0})}}{(\omega - i/RC)(\omega - \omega_0)}. \quad (22)$$

$$I_3 = \int \frac{d\omega e^{i\omega t}}{(\omega - i/RC)(\omega_0 + \omega)}. \quad (23)$$

$$I_4 = - \int \frac{d\omega e^{i\omega(t - \frac{2N\pi}{\omega_0})}}{(\omega - i/RC)(\omega + \omega_0)}. \quad (24)$$

The solution will be obtained now for two different domains, but all the machinery for evaluating contour integrals carries over, making the suitable contour, shifting the poles upward to ensure a causal solution, finding the residues, and so on.

1. **For**  $0 \leq t \leq \frac{2N\pi}{\omega_0}$ ,

We observe for this case that only (21) and (23) contribute to the integral when the semicircular contour is closed in the upper half plane, since no poles exist in the lower half plane.

Therefore evaluation of (21) and (23) is sufficient to solve the problem. If we do so, using the methods elucidated with the previous examples (albeit with a little frustrating algebra), we



would obtain the following as a solution:

$$q = \frac{V_0/R}{\omega_0^2 + (\frac{1}{RC})^2} \left( -\omega_0 \cos \omega_0 t + \frac{1}{RC} \sin \omega_0 t + \omega_0 e^{-t/RC} \right). \quad (25)$$

The exponential term in (25) is a rapidly dying term so eventually the solution can be approximated for large enough  $t$  as,

$$-\frac{V_0 \cos(\omega_0 t + \theta)}{R \sqrt{\omega_0^2 + (\frac{1}{RC})^2}}. \quad (26)$$

This implies that the voltage across the capacitor:

$$V_C = \frac{q}{C} = -\frac{V_0 \cos(\omega_0 t + \theta)}{RC \sqrt{\omega_0^2 + (\frac{1}{RC})^2}}, \quad (27)$$

Where  $\theta = \tan^{-1} \frac{1}{\omega_0 RC}$ .

This shows that as the frequency is increased, the amplitude of the signal goes on decreasing. For large  $\omega_0$  the amplitude tends to 0. That is precisely the functioning of the **low-pass filter**. It allows for the amplitude to be large when the frequency is small, but reduces the amplitude to 0, for large input frequencies.

Let us now consider the voltage across the resistor.

$$\begin{aligned} V_R &= R \frac{dq}{dt} \\ &= \frac{V_0 \omega_0}{\omega_0^2 + (\frac{1}{RC})^2} \left( \omega_0 \sin \omega_0 t + \frac{1}{RC} \cos \omega_0 t - \frac{1}{RC} e^{-t/RC} \right). \end{aligned} \quad (28)$$

In the limit of sufficiently large time  $t$ , the exponentially decaying term is eliminated and we are left with,

$$\begin{aligned} V_R &= \frac{V_0 \omega_0}{\omega_0^2 + (\frac{1}{RC})^2} \left( \omega_0 \sin \omega_0 t + \frac{1}{RC} \cos \omega_0 t \right) \\ &= \frac{V_0}{1 + (\frac{1}{RC \omega_0})^2} \left( \sin \omega_0 t + \frac{1}{RC \omega_0} \cos \omega_0 t \right) \\ &= \frac{V_0 \cos(\omega_0 t - \phi)}{\sqrt{1 + (\frac{1}{RC \omega_0})^2}}, \end{aligned} \quad (29)$$



Where  $\phi = \tan^{-1}(\omega_0 RC)$ .

This tells us how the RC circuit acts as a **high-pass filter** when the voltage is taken across the resistor. If the frequencies are high, the amplitude is high; if low, the amplitude is low.

2. **For  $t > \frac{2N\pi}{\omega_0}$ :**

All of the integrals (21) to (24) contribute to the final result. If worked out carefully, we find that the sinusoidal parts cancel and we are left with only two exponentially decaying terms, as is expected once the power is switched off.

$$\begin{aligned}
 q &= \frac{V_0\omega_0/R}{\omega_0^2 + (\frac{1}{RC})^2} \left[ \exp\left(\frac{-t}{RC}\right) - \exp\left(-\frac{(t - \frac{2N\pi}{\omega_0})}{RC}\right) \right] \\
 &\approx \frac{V_0\omega_0/R}{\omega_0^2 + (\frac{1}{RC})^2} \exp\left(\frac{t - 2N\pi/\omega_0}{RC}\right). \quad (30)
 \end{aligned}$$

The second term is valid for large  $t$ , which is the case if the signal is switched off after a long time.

## 4. Integration and Differentiation Using RC Circuits

### 4.1 The Integrator

The above RC circuit in the limit of **large time constant** can work as an integrator when the voltage is measured across the capacitor. Let us see how it integrates  $V_{in}(t)$  using the theory of FTs.

We can write the inversion(10) as,

$$\begin{aligned}
 q &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{V}_{in}C}{1 + i\omega RC} e^{i\omega t} d\omega \\
 &= \frac{1}{iR\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{V}_{in}}{\omega - \frac{i}{RC}} e^{i\omega t} d\omega. \quad (31)
 \end{aligned}$$



Now, let us invert the FT and write,

$$\begin{aligned}
 V_C = \frac{q}{C} &= \frac{1}{2i\pi RC} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt' V_{in}(t') e^{-i\omega t'} \frac{e^{i\omega t}}{\left(\omega - \frac{i}{RC}\right)} d\omega \\
 &= \frac{1}{2i\pi RC} \int_{-\infty}^{\infty} dt' V_{in}(t') \int_{-\infty}^{\infty} \frac{e^{i\omega(t-t')}}{\left(\omega - \frac{i}{RC}\right)} d\omega.
 \end{aligned}
 \tag{32}$$

The  $\omega$  integral (let's call it  $I$ ) was previously encountered while working out the response for the delta function input. It simply gives us the following result.

$$\begin{aligned}
 I &= 2\pi i e^{\frac{-(t-t')}{RC}} \quad (t' \leq t) \\
 &= 0 \quad (t' > t).
 \end{aligned}
 \tag{33}$$

But since  $RC \rightarrow \infty$ , the solution becomes:

$$\begin{aligned}
 I &= 2\pi i \quad (t' \leq t) \\
 &= 0 \quad (t' > t).
 \end{aligned}
 \tag{34}$$

Putting the above back into the expression for  $V_C$  (and keeping in mind that for  $t' > t$ ,  $I = 0$ ), we have,

$$\begin{aligned}
 V_C &= \frac{1}{RC} \int_{-\infty}^t V_{in}(t') dt' \\
 &= \frac{1}{RC} \int_0^t V_{in}(t) dt.
 \end{aligned}
 \tag{35}$$

The limits on the integral can be changed since  $V_{in} = 0 \forall t < 0$ . The above exercise convinces us how the RC circuit works as an integrator!

#### 4.2 The Differentiator

The differentiator can be dealt with by taking the voltage across the resistor R, **but in the limit of small RC**, i.e,  $RC \rightarrow 0$ .



For small  $RC$ , we have,

$$\begin{aligned} q &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{V_{in} C}{1 + i\omega RC} e^{i\omega t} d\omega \\ &\approx \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{V}_{in} e^{i\omega t} d\omega. \end{aligned} \quad (36)$$

Now,

$$\begin{aligned} V_R &= R \frac{dq}{dt} \\ &= RC \frac{d}{dt} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{V}_{in} e^{i\omega t} d\omega \right) \\ &= RC \frac{dV_{in}}{dt}. \end{aligned} \quad (37)$$

Precisely what we set out to show!

## 5. Conclusion

This method of solving inhomogeneous differential equations using homogeneous boundary conditions is illuminating because it is the most general approach to the problem. Although we considered very specific potentials here, but any potential can be essentially thought of as a superposition of sinusoids or impulse functions with proper weights, and since the equation is linear in  $q$ , the solutions can be obtained by adding up these individual components together.

The problem itself is of use in various aspects of network analysis, and in some cases, the modelling of some important natural phenomena that involve charging and discharging through resistive media. Many other differential equations in network analysis, such as those of LR circuits, RLC circuits can be solved quite simply using the methods discussed. Although a very simple problem in itself, solving the RC circuit using this method grants a lot of insight into contour integration (such as shifting of singularities that lie on the contour itself to maintain causality), as well as into some of the fundamentals of why some electronic circuits work the way they do.



## 6. Acknowledgement

The work done here was inspired by Dr Bikram Phookun, Associate Professor, St. Stephen's College, University of Delhi. The plots which are there in the article were very kindly constructed for me by my friend Abhishek Chakraborty, who is currently a student of 2nd year Physics at St. Stephen's College, University of Delhi.

## Suggested Reading

- [1] Arfken, Weber and Harris, *Mathematical Methods for Physicists*, Elsevier publications, Chapter 11 - Complex Variable Theory and Chapter 20 - Integral Transforms, 7th Edition, 2012.

