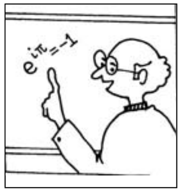

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

A 1235711 Petalled Flower

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This article contains very simple mathematics and a lot of creativity. Here, we will sketch a 3-dimensional flower-like structure, not using complicated equations but geometric ideas instead. The speciality of 1235711 will be mentioned. At first we will study a general situation in 3-dimensional space $\mathbb{R}^3$ and then we will take up the flower as an example of the general case.

Consider the 3-dimensional space $\mathbb{R}^3$ and fix a point as the origin. We draw the three mutually perpendicular axes X, Y and Z and let M be the set of all possible rays with vertex at the origin. Here, a straight line l in $\mathbb{R}^3$ is said to be a ray if and only if for every plane containing l there exists a point $O \in l$ and $r \in \mathbb{R}^+$ such that the circle with centre O and radius r in the plane containing l, intersects l at exactly one point.

Observe that M forms a partition of $\mathbb{R}^3 \setminus \{\text{the origin}\}$. Thus for $P(x, y, z) \in \mathbb{R}^3$ which is not the origin, there exists R_{OP} (or simply R) $\in M$ such that $P \in R$.

Now the XY-plane contains a point $P'(x, y, 0)$ which is the projection of P on the XY-plane; join OP'. Then $\angle P'OX$ is the angle which OP' subtends with the positive X-axis. For the point P we define its two characteristics as follows: the characteristic Y-angle of P (to be denoted by $\angle_y P$) is $\angle P'OX$ and the characteristic

Keywords
Geometry, sets, rays, spirals, prime numbers.

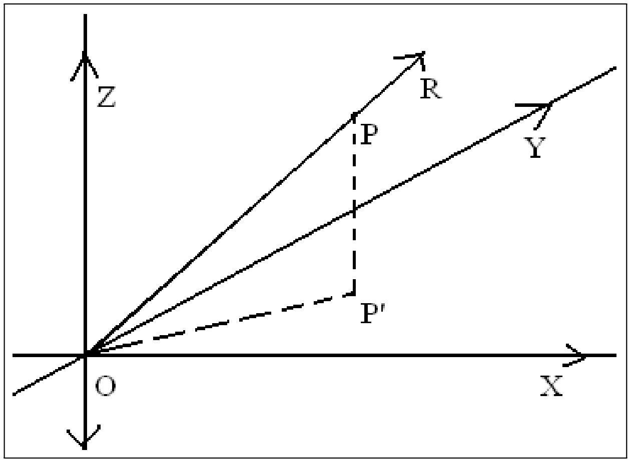


Figure 1. Characteristic angles of P.

Z-angle of P (to be denoted by $\angle_z P$) is $\angle P'OP$. (Refer *Figure* 1).

Thus,

$$-\pi/2 \leq \angle_z P \leq \pi/2 \text{ and } 0 \leq \angle_y P \leq 2\pi.$$

However, we can also use $-2\pi \leq \angle_y P \leq 0$, depending on our purpose.

Proposition 1: For $R \in M$ and $P, Q \in R^* = R \backslash \{O\}$,

$$\angle_y P = \angle_y Q \text{ and } \angle_z P = \angle_z Q$$

Characteristic angles are unique features of rays in the 3-dimensional space.

The proof is simple and we leave the details to the reader. Thus every point on R^* has the same characteristic angles. Thus we may represent the ray R by the ordered pair $(\angle_y R, \angle_z R)$ where $\angle_y R = \angle_y P$ and $\angle_z R = \angle_z P$, for $P \in R^*$.

Proposition 2: For $K, R \in M$,

$$\angle_y K = \angle_y R \text{ and } \angle_z K = \angle_z R$$

if and only if $K = R$.

Proof: Let $R, K \in M$ and $P \in R^*, Q \in K^*$. Now

$$\angle_y R = \angle_y K \Rightarrow \angle_y P = \angle_y Q$$

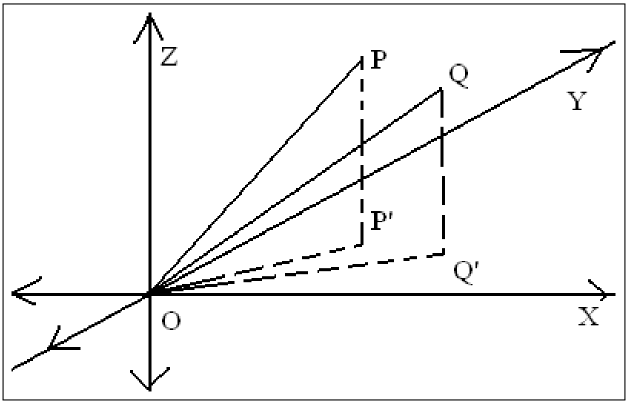


Figure 2. P and Q and their projections P′ and Q′ respectively in the 3-dimensional space.

P' and Q' are the projections of P and Q on the XY-plane (Refer *Figure* 2). Thinking from the perspective of polar coordinates, it follows that

$$\angle P'OX = \angle Q'OX \Rightarrow OP' = k\ OQ',\ k \in \mathbb{R}^+.$$

Thus P' and Q' lie on the same ray which implies that both P and Q lie on the same plane (say $ABCD$) bounded by the Z-axis. The plane $ABCD$ intersects the XY-plane at OB' (Refer *Figure* 3). Now we focus on the plane ABCD (Refer *Figure* 4).

We have $\angle_z K = \angle_z R$ which gives $\angle_z P = \angle_z Q$. By using polar coordinates once again,

$$\angle_z P = \angle_z Q \Rightarrow \angle POB' = \angle QOB' \Rightarrow OP = k'\ OQ,\ k' \in \mathbb{R}^+$$

which shows that P, Q lie on the same ray. Hence $R = K$. The converse is clear.

Thus each $R \in M$ has unique characteristic angles. Now let $D \subset \mathbb{R}^2$ be a domain and $f : D \to \mathbb{R}$ a function. Let

$$J = \{R \in M \mid f(\angle_y R, \angle_z R) = 0\}.$$

Here, f can be a polynomial function, an exponential function etc. For a given f we will get a corresponding shape in $\mathbb{R}^3$. In fact, with the help of characteristic angles we can draw any plane curve on any symmetrical figure we want.

Any plane curve can be drawn on any symmetrical figure in the 3-dimensional space.

Figure 3. Plane ABCD containing both P and Q.

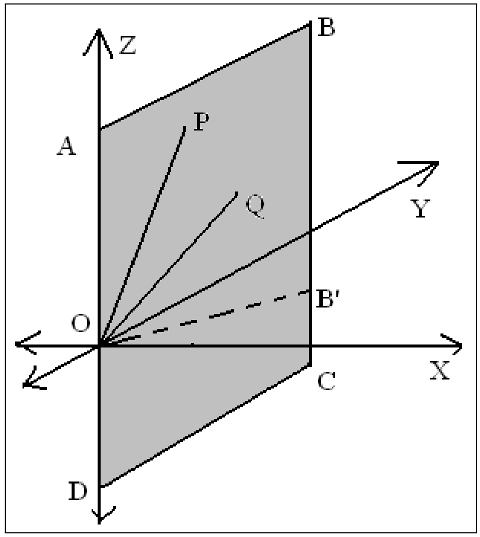
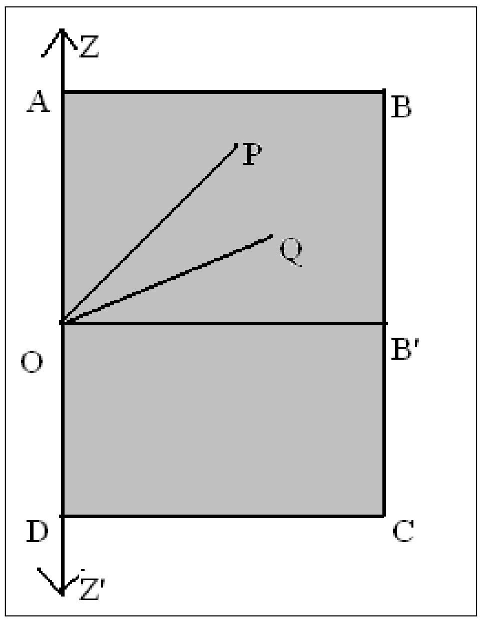


Figure 4. Plane ABCD viewed from a clearer perspective.



This method is illustrated in the following steps:

Consider a plane curve in the YZ-plane defined by $g(y, z)$=0, let

$$G_g = \{(y, z) \in \mathbb{R}^2 \mid g(y, z) = 0\}.$$

Choose $r \in \mathbb{R}$ and let

$$G = \{(x, y, z) \in \mathbb{R}^3 \mid x = r \text{ and } (y, z) \in G_g\}.$$

Thus in $\mathbb{R}^3$, G is a curve on the plane $x = r$. Let $P(r, y, z) \in G$ and P' be the point $(r, x, 0)$. We join OP' and $P'A$ where A is $(r, 0, 0)$. Consider the ray R with vertex at O and passing through P. Then

$$\angle_y R = \tan^{-1}(\frac{|P'A|}{|OA|}) = \tan^{-1}(\frac{|P'A|}{|r|}).$$

Also $|P'A|^2+r^2 = |OP'|^2$ which implies that $|OP'| = \sqrt{|P'A|^2 + r^2}$. Thus,

$$\angle_z R = \tan^{-1}\left(\frac{|PP'|}{|OP'|}\right).$$

See *Figure* 5 below.

Since we have the characteristic angles, we can uniquely locate R. Thus corresponding to each $P \in G$, we can uniquely locate $R \in M$ such that $P \in R$. Observe that no two $P, Q \in G$ lie on the same ray and hence for every point $P \in G$ we determine the corresponding $R \in M$. Let J' be the set of all such R. Consider a symmetrical figure $\mathcal{H}$ and let O' be a point on its line of symmetry. Now we place the figure $\mathcal{H}$ in $\mathbb{R}^3$ with O' at origin. Then the points which belong to $\mathcal{H} \cap J'$ will give the curve defined by $g(y, z)$ on $\mathcal{H}$.

We've said that various types of f give rise to various types of shapes in $\mathbb{R}^3$. Let us exploit the freedom in choosing f. Define

$$f : [0, 2\pi] \times [0, \frac{\pi}{2}] \to \mathbb{R}$$

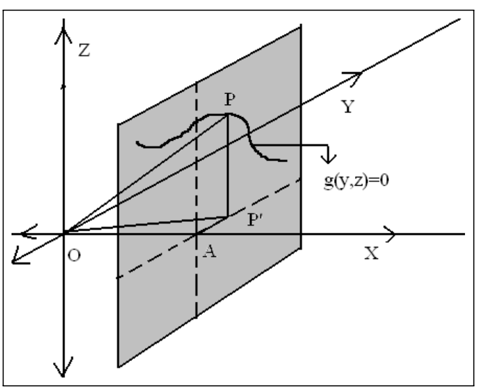


Figure 5. The curve $g(y, z) = 0$ on the plane x=r′.

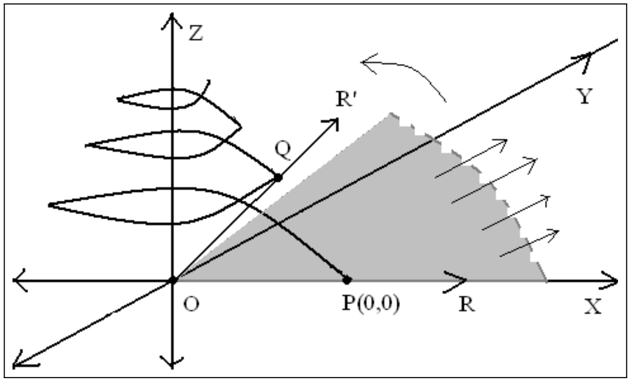


Figure 6. The spiral sheet J is shaded in gray.

Polar and Cartesian systems are just two representations of the same collection of points.

by

$$f(\angle_y R, \angle_z R) = \angle_y R - n\angle_z R,\ n \in [4, \infty)$$

Here, $J = \{R \in M \,|\, \angle_y R = n\angle_z R\}$. Geometrically, J will be a spiral sheet (shaded in *Figure* 6), extending to infinity, that is centered at the origin and which spirals around the Z-axis; see *Figure* 6. So for $\angle_z R = 0 \Rightarrow \angle_y R = 0$ which refers to the ray R_{OP} (shown in *Figure* 6) and for $\angle_z R = \frac{2\pi}{n} \Rightarrow \angle_y R = 2\pi$ which refers to the ray R_{OQ} (shown as R' in *Figure* 6)

For a given $a > 0$, consider the curve

$$r = a(1 + |\cos 4\theta|)\,,$$

which is written in polar coordinates. Note that $r_{max} = 2a$ and $r_{min} = a$. This curve is shown in *Figure* 7 which sweeps out an eight-petalled flower.

Let $G_a = \{(r, \theta) \in \mathbb{R}^2 \,|\, r = a(1 + |\cos 4\theta|)\}$. Corresponding to every $(r, \theta) \in G_a$ there is a unique $(x, y) \in \mathbb{R}^2$. Let

$$S'_a = \{(x, y) \in \mathbb{R}^2 \,|\, (x, y) \text{ corresponds to} (r, \theta) \text{ in } G_a\}.$$

Clearly $S'_a = G_a$ as they contain the same set of points. But for a point, say $T \in S'_a$ and G_a, T has a cartesian representation (x, y) (as $T \in S'_a$) and a polar representation (r, θ) (as $T \in G_a$), and (x, y) may not be equal to (r, θ), i.e., the two representations may not be the same.

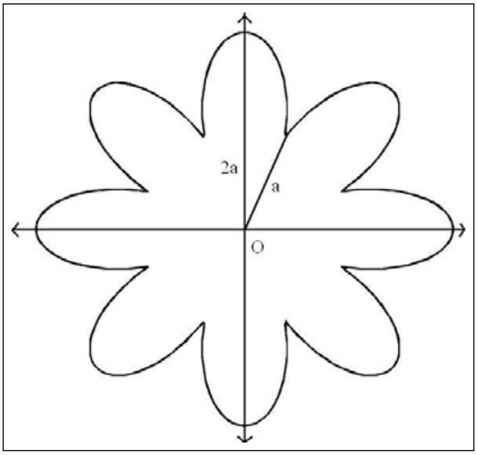


Figure 7. Graph of the equation $r = a(1 + |\cos 4\theta|)$.

Now let us work in $\mathbb{R}^3$. Let

$$S_a = \{(x, y, z) \in \mathbb{R}^3 \,|\, (x, y) \in S_a^{'} \text{ and } z = 1 - a\}.$$

where a varies in the interval $[0, 1]$. We interpret the sets S_a geometrically – see *Figure* 8. S_1 will be an eight-petalled flower in the XY-plane. Indeed, by definition,

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, (x, y) \in S_1^{'} \text{ and } z = 0\}.$$

Observe that $S_0 = \{(0, 0, 1)\}$. Let

$$S = \bigcup_{a \in [0,1]} S_a.$$

Geometrically, S will look like a cone with height 1 unit and whose central axis is along the Z-axis and with a wavy surface

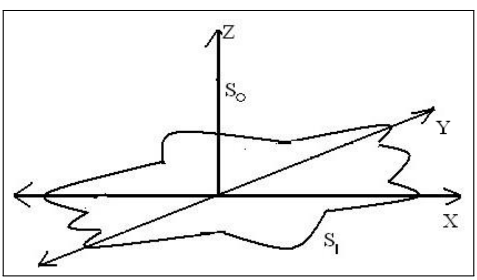


Figure 8. Geometrical interpretation of the sets S_a.

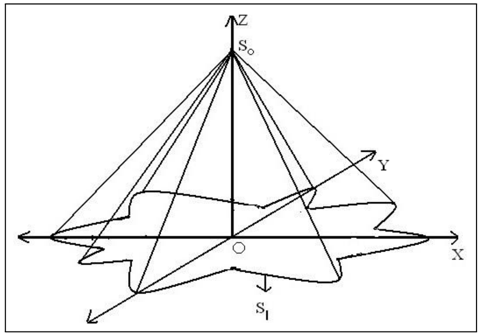


Figure 9. Geometrical interpretation of set S.

We are seeing only $\mathcal{B}$ which now looks like a 1235711 petalled flower.

with eight-petal like deformations (Refer *Figure* 9) and S_1 is the last flower in S. Now recall that centered at the origin and spiralling along the Z-axis is a spiral sheet J. As the centre of S_1 is the origin O (in this case) and as the centre of S_1 lies in the line of symmetry of S so, geometrically $\mathcal{B} = J \cap S$ will be a spiral, spiralling around S with 8 petals on each spiral.

Now recall that $J = \{R \in M \mid \angle_y R = n\angle_z R\}$. A single spiral corresponds to a total variation from 0 to 2π in the value of $\angle_y R$ as shown in fig 6. Thus the total number of spirals that we obtain is

$$m = (\frac{\pi}{2}) \div (\frac{2\pi}{n}) = \frac{n}{4}.$$

This is why the interval $[4, \infty]$ is so chosen, otherwise we wouldn't have got any spiral. So $\mathcal{B}$ has $n/4$ spirals around S with 8 petals on each spiral and hence $\mathcal{B}$ has a total of $2n$ petals around S.

Now suppose we choose,

$$J = \{R \in M \mid \angle_y R = 617855.5\angle_z R\}.$$

Then $\mathcal{B}$ will have in total $(2\times617855.5) = 1235711$ petals around S. Now we close our eyes and imagine ! We imagine S with $\mathcal{B}$ around it. Now we forget S only! What do we see ? We are seeing only $\mathcal{B}$ which now looks like a 1235711 petalled flower.

Conclusion

This example illustrates how different choices of f give rise to beautiful shapes using characteristic angles. The number 1235711 is not an arbitarily choosen number. It is the smallest prime number that can be made by juxtaposing primes and putting 1 infront of it. The next such prime number is 123571113171923.

Acknowledgement

I would like to thank the reviewer for giving valuable comments which helped me enhance the quality of the article.

Suggested Reading

[1] **Serge Lang, Gene Murrow, *Geometry*, Springer.**

[2] **Edwin E Moise, *Elementary geometry from an advanced standpoint*, Addison-Wesley Publishing Company, INC.**