
Scaling Concepts in Describing Continuous Phase Transitions

Srikanth Sastry

Phase transitions, like the boiling of water upon increasing temperature, are a part of everyday experience and are yet, upon closer inspection, unusual phenomena, and reveal a host of fascinating features. Comprehending key aspects of phase transitions has led to the uncovering of new ways of describing matter composed of large numbers of interacting elements, which form a dominant way of analysis in contemporary statistical mechanics and much else. An introductory discussion is presented here of the concepts of scaling, universality and renormalization, which forms the foundation of the study of continuous phase transitions, such as the spontaneous magnetization of ferromagnetic substances.

Overview

Substances around us exist in various forms, and we are used to the idea that changes in their environment lead to changes in their properties. When a liquid is heated, it expands, or occupies greater volume, a phenomenon we are familiar with in using a thermometer. We are also familiar with the possibility that we can pack in greater amounts of gas into roughly the same volume, or increase its density, by applying pressure, as in pumping a bicycle tube or blowing a balloon. The responses of the substance in question are gradual – the increase in volume occupied by mercury in a thermometer is proportional to the change in temperature, or the increase in the density of air in a tube or balloon is proportional to the pressure we exert. These proportionalities could be expressed by numbers, called the coefficient of thermal expansion, and the compressibility. An equally familiar, but dramatically different change occurs when water is heated up to a temperature of 100° C from room temperature at atmospheric pressure. Initially, if we were to carefully observe, a rise in temperature leads to an increase in volume. But a small change



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Phase transitions, critical point, scaling laws, universality, renormalization.



from below 100°C to just above, the volume of water changes dramatically, by roughly 1600 times and it transforms to vapour, an instance of a *phase transition* that we are perhaps most familiar with. Here, the change in the density with temperature is not smooth, but *singular*, displaying a discontinuity, as shown in *Figure 1*. The conversion of water from the liquid state to vapour is an instance of a *liquid-gas* transition, which at atmospheric pressure is a discontinuous transition, referred to as a *first order* transition.

If water is heated not at atmospheric pressure, but at a higher pressure, the transition to the vapour phase occurs at a higher temperature. The drop in density at the transition becomes smaller as the pressure is increased, till at a pressure of 218 atmospheres, the density no longer has a discontinuous drop, although the change is precipitous, with the slope of change being infinite, at a temperature of 374°C. At higher pressures, the density change is smooth, and there is no longer a clear distinction to be made between liquid water and vapour. The end point of the line of discontinuous transitions, at the pressure of 218 atmospheres and temperature of 374°C, is called a *critical point*, and marks the point below which liquid and gas phases emerge as distinct states in which water can exist. It is a point at which the transition from one state to the other is continuous.

The behaviour near such a continuous transition point displays many remarkable features, which forms the subject of this article. A central feature relates to the cessation of the distinction between liquid and gas at the critical point. Close to the critical point, the density becomes increasingly inhomogeneous in space, with the density being higher than the average in some places and lower in others. One may describe the situation as arising from the system getting confused as to which state to be in, as the distinction between liquid and gas becomes blurred, and moving from one state to the other becomes correspondingly easier. Such inhomogeneities are apparent if we measure local densities in sub-volumes that are smaller than a characteristic size or length scale that depends on of how far we are from the critical point. An experimental measurement that probes such density variations is light scattering, which reveals strong scattering of light near a critical point, when the length scale of density inhomogeneities becomes comparable to the wavelength of light, in a striking

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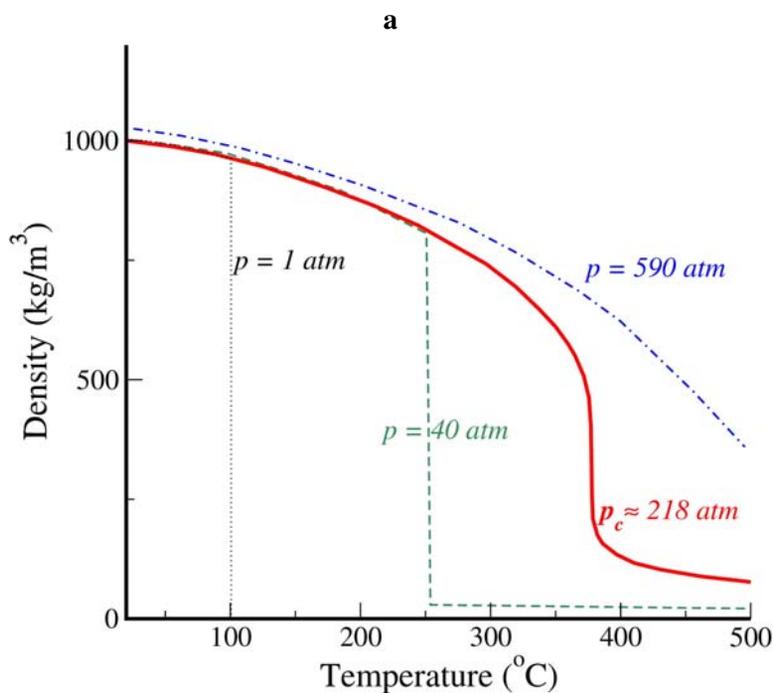
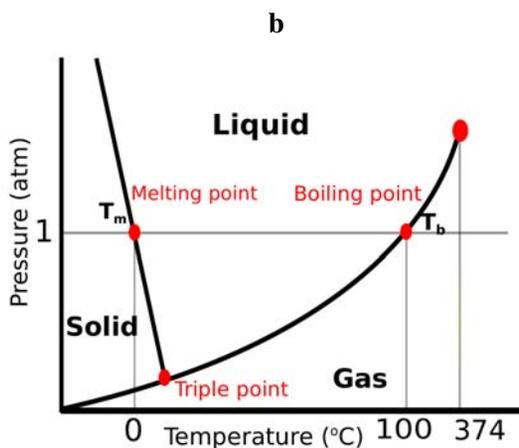


Figure 1. (a) Density of water vs. temperature and pressure. (b) The pressure temperature phase diagram of water (not to scale). (Part (b) adapted from: <http://www.kentchemistry.com/links/Matter/Phasediagram.htm>).

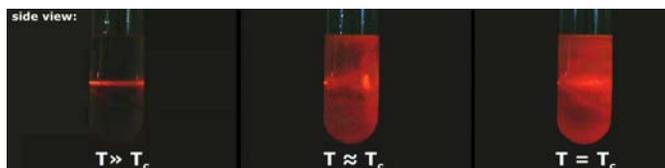


phenomenon known as ‘critical opalescence’ (Figure 2¹). How

¹The inhomogeneities in this demonstration are of composition in a mixture of two liquids rather than density.

Figure 2. Critical Opalescence – a phenomenon wherein light gets strongly scattered because of density or composition fluctuations near a critical point, is seen here in a liquid mixture of Aniline and Cyclohexane.

(From: <http://www.doitpoms.ac.uk/lplib/solid-solutions/demo.php>)



large does the length scale of density fluctuations grow? The amazing answer, comprehending which has prompted the development of new approaches we describe here, is that the growth of this length scale is unbounded. As the critical point is approached, density fluctuations occur on larger and larger scales, and precisely at the critical point, we cannot assign a finite length scale to the density fluctuations – they occur on all length scales.

The fact that fluctuations occur on length scales that are unbounded (or ‘diverge’) as the critical point is approached suggests that perhaps the details of the composition of a substance microscopically (such as the exact nature of interactions between atoms or molecules) should somehow become unimportant for at least some features of behaviour seen near the critical point. This expectation is indeed met to a surprising degree by experimental evidence. *Figure 3* shows experimental data for the densities of coexisting liquid and gas phases, for a range of temperatures below the critical temperature, and for a wide range of substances. In each case, the densities and temperatures have been scaled to their values at the critical point, resulting in a remarkable ‘data collapse’ – regardless of the substance, the densities fall along the same curve, even while the critical temperatures and densities vary considerably between these substances. At first glance, the observed data collapse may seem like supporting evidence for the law of ‘corresponding states’ from the van der Waals equation of state, familiar from elementary treatments of phase transitions. This is true in the limited sense that the observed data superimpose with the scaling suggested by van der Waals theory. However, the vanishing of the difference in density between the two phases as the critical temperature is approached is, while the same for all substances, different from the behaviour predicted by van der Waals theory. In *Figure 3*, the curve that is drawn through the data is a fit by the notable physical chemist E A Guggenheim,

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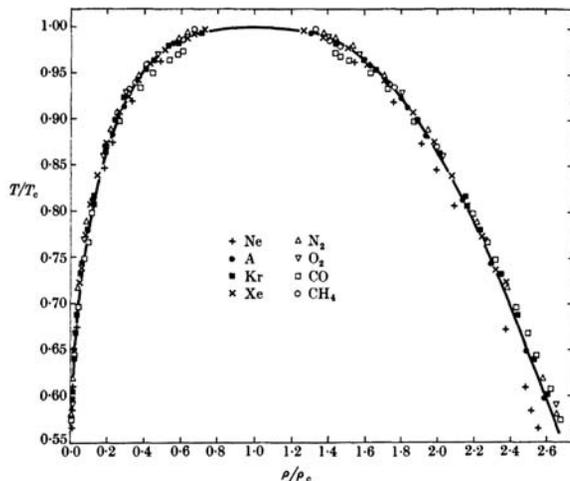


Figure 3. Densities of coexisting liquid and gas phases of a variety of substances, plotted against temperature, with both densities and temperatures scaled to their value at the critical point, resulting in a striking superposition of all data on to a single curve regardless of the substance. (From: E A Guggenheim, *J. Chem. Phys.*, Vol.13, 253, 1945)

which has the density difference vanishing as the distance to the critical temperature raised to a power $1/3$, clearly distinct from $1/2$, the ‘mean field’ value predicted by so-called classical theories like van der Waals theory.²

Two key features of critical phenomena are illustrated by the data discussed. The first is the observation of ‘universality’, a property that the critical behaviour of diverse substances displays features that are the same, or universal. In our example, the set of distinct systems were different fluids, studied near their liquid-gas critical point. The degree of universality of behaviour in this class is already quite impressive, but there is more. Entirely different physical systems, such as some ferromagnets, binary alloys, liquid mixtures, all exhibit critical behaviour that can be described in the same way. These systems are described as belonging to the ‘Ising universality class’ in three dimensions (as explained further later on). Different phase transitions belong to different universality classes that are broadly determined by the spatial dimensionality, and the nature of quantities that describe the kind of

²Mean field values of exponents are also predicted from Landau theory, an important theoretical approach which we shall not discuss in this article. An interesting discussion of Landau theory is found in a ‘Classroom’ article by Rekha S Raorane *et al*, The Landau Theory of Phase Transitions: A Mechanical Analog, *Resonance*, Vol.14, No.7, pp.704-713, 2009.



ordering involved in the system undergoing the phase transition. The latter (e.g. the density) are, for obvious reasons, called *order parameters*, and play a significant role in discussions of phase transitions. The other key feature is that each universality class is characterised by a set of numbers, called ‘critical exponents’, which are unique for that universality class, and are non-trivial, in the sense that they cannot be deduced by simple considerations like dimensional analysis or from theories such as the van der Waals theory which assume smooth behaviour near the critical point. We shall elaborate on this point later. It turns out that the critical exponents describing the behaviour of different quantities are interrelated and not all independent. Attempts to understand these relations was among the main motivations that lead to scaling descriptions of behaviour near critical points.

Phase Transitions: Preliminaries

The theoretical analyses of critical points, as well as their exposition, have employed simplified models of interacting systems. We follow the same practice here, and define here the Ising model and provide definitions of essential quantities of interest later. This section may thus be skipped on first reading by those familiar with the Ising model and consulted as needed. Those who are not familiar with the definitions here need only accept the statement that the partition function as defined provides an expression for the free energy, and the remaining statements can be obtained from thermodynamic definitions.

Employing simple models to analyse critical behaviour is justified, *a posteriori*, based on the understanding that the critical behaviour of different systems is the same, if they belong to the same *universality class*. Keeping this in mind, we might attempt to develop a simplified model to study the liquid-gas transition. The basic requirement for the liquid-gas transition is that one has a system of particles which have attractive interactions between them. The strength of interactions determine a temperature below which attractions become relevant and some form of *condensation* may be expected. A minimal model to study the liquid-gas transition is the *lattice gas* model, which can be defined as follows: (i) We divide space into a number of cells, such that each cell can at the most be occupied by one particle, and it is possi-



ble for adjacent cells to be occupied. The ‘occupancy’ of a cell is given by a variable n_i which is 0 if the cell is empty or 1 if the cell is occupied. (ii) If two adjacent sites are occupied, there is an attractive interaction of energy $-\epsilon$. We can thus write the interaction energy, or the *Hamiltonian*, of this system as,

$$H_{LG} = -\epsilon \sum_{\langle ij \rangle} n_i n_j, \quad (1)$$

where i and j indicate spatial positions, and $\langle ij \rangle$ indicates all adjacent site pairs or ‘nearest neighbour’ pairs.

Without further discussion of the lattice gas model, we note that this model has strong resemblance to another commonly studied model, the Ising model, which is a simplified model for the paramagnetic to ferromagnetic transition in magnetic systems. In this model, one has spin variables, S_i , at lattice sites i , which can take values of +1 or -1. The Hamiltonian for this system, in the presence of a magnetic field, h , is,

$$H_{Ising} = -J \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i, \quad (2)$$

where J measures the strength of the interaction between spins. As before, i and j indicate spatial positions, and $\langle ij \rangle$ indicates all adjacent site pairs or nearest neighbour pairs. In discussing specific results, we consider this model for spatial dimension $d = 3$, although illustrations, including results from computer simulations shown later, will be for $d = 2$.³

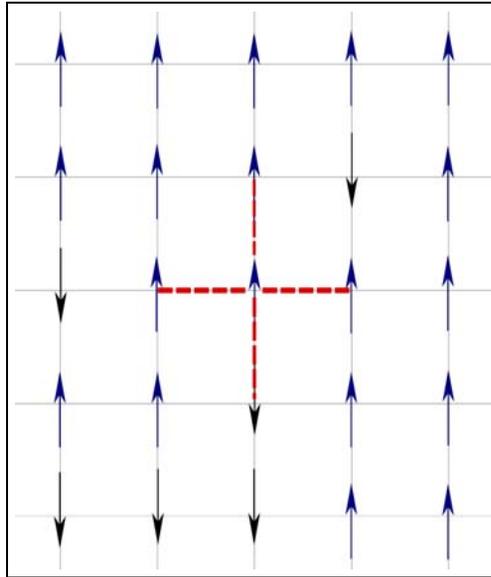
An inspection of the interaction energies for the two models reveals that the two models are similar. Indeed, we can map one to the other exactly, by considering $S_i = 2n_i - 1$. We shall not discuss the lattice gas model further here, but proceed further by considering the Ising model.

A microscopic description of a state of the Ising model can be given in terms of a *spin configuration*, for which we may compute

³Real physical systems of course exist in 3D-space, but effectively lower dimensions can arise, for example, if the interactions between spins along one axis are close to zero, leading to a 2D-spin system. Theoretically, understanding critical behaviour by including a change of spatial dimensionality forms an effective, and in many ways essential, tool.



Figure 4. Schematic of the Ising model defined on a two dimensional square lattice. The dashed red lines indicate the nearest neighbour interactions for the central spin.



the energy. The spin configuration, which we denote henceforth by $\{S\}$, specifies the value of each spin S_i at each site. A summation of the Boltzmann weight for each configuration, over all spin configurations, results in the *partition function*, from which the free energy can be obtained. For the Ising model, we have,

$$\begin{aligned} Z_{Ising} &= \exp(-\beta F) = \sum_{\{S\}} \exp[-\beta H_{Ising}] \\ &= \sum_{\{S\}} \exp[-\beta \{-J \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i\}]. \end{aligned} \quad (3)$$

A schematic representation of the Ising model is given in *Figure 4*.

The principal quantity of interest in studying the transition to a ferromagnetic state is the magnetization, which is given in terms of the free energy as,

$$M = - \left(\frac{\partial F}{\partial h} \right)_T = \left\langle \sum_i S_i \right\rangle, \quad (4)$$

where (here and throughout) $\langle (\dots) \rangle = \frac{1}{Z} \sum_{\{S\}} (\dots) \exp[-\beta H_{Ising}]$ is the ensemble average. In addition to the magnetization, *response*

functions, (i) the heat capacity C_H which measures how the internal energy U of the system changes with temperature, and (ii) the susceptibility χ , which measures the change of magnetization with the field, are of interest, and are given by,

$$C_H = \left(\frac{\partial U}{\partial T} \right)_h = -T \left(\frac{\partial^2 F}{\partial T^2} \right)_h, \quad (5)$$

and

$$\chi = \frac{1}{V} \left(\frac{\partial M}{\partial h} \right)_T = \frac{-1}{V} \left(\frac{\partial^2 F}{\partial h^2} \right)_T. \quad (6)$$

The susceptibility, which we define here per unit volume, can be written explicitly as,

$$\chi = \frac{1}{Vk_B T} \left[\left\langle \sum_{i,j} S_i S_j \right\rangle - \left\langle \sum_i S_i \right\rangle^2 \right], \quad (7)$$

which indicates that the ‘correlation’ between values of spins at different locations determines the susceptibility. By defining local magnetization $m(\mathbf{r})$ (which we may consider to be the magnetization defined over small volumes of space) and going over to integrals over spatial coordinates in place of sums over lattice positions, we may write,

$$\chi = \frac{1}{k_B T} \int d^d r G(r), \quad (8)$$

where the ‘correlation function’ is defined as

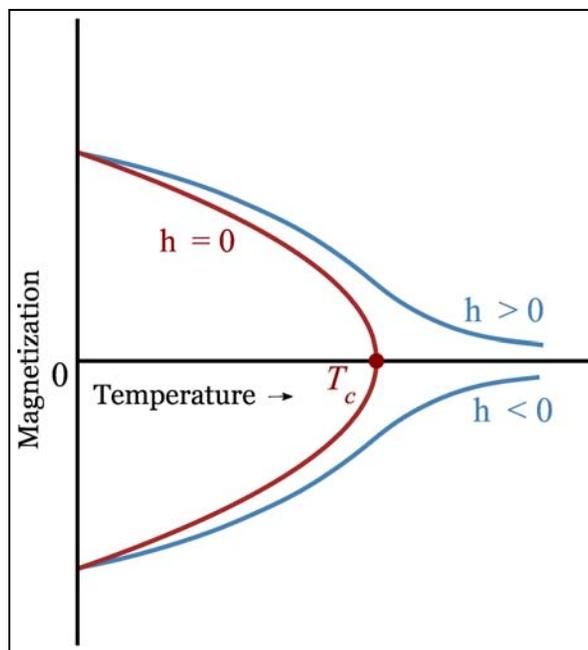
$$G(r) = \langle m(\mathbf{r})m(\mathbf{0}) \rangle - \langle m(\mathbf{0}) \rangle^2. \quad (9)$$

Here we have made a few assumptions, namely that the correlation function between magnetizations at two positions depends only on the scalar distance between them, and not on either the direction, or the specific coordinates of the two points in space (hence one of the positions can be treated as the origin of the coordinate system, and $G(\mathbf{r}-\mathbf{r}') = G(r)$, where $r = |\mathbf{r}-\mathbf{r}'|$). We may expect that if the spins are correlated in space, there should be a characteristic distance over which these correlations are present. This characteristic distance or ‘correlation length’ is denoted by ξ .

The quantities defined above enable us to study the manner in which the Ising system behaves in the vicinity of the critical point.



Figure 5. (a) Magnetization as a function of temperature and magnetic field in the Ising model. (b) The temperature-field phase diagram



Critical Exponents and the Scaling Hypothesis

The magnetization in the Ising model as a function of temperature and applied magnetic field is shown in *Figure 5*. There are a few points to note before we proceed to discuss it. The Ising model, by its definition, is symmetric with respect to a change of signs of all the spins, along with a change in sign of the applied magnetic field. Thus, at zero temperature, and zero field, it has two equivalent ground states, one with all the spins = +1 and another with all the spins = -1. At finite temperatures and fields, one continues to observe the \pm symmetry as stated, which is a simplifying feature of this model and the uniaxial magnetic systems it describes. At finite applied field, the magnetization displays ‘paramagnetic’ behaviour, rising from a value of zero at very high temperatures, to a (per site) value of ± 1 at zero temperature. The critical value of the applied magnetic field h_c is 0. In this case, by the symmetry mentioned above, one should expect the magnetization to be zero at all temperatures. This is certainly true at high temperatures. However, when the temperature is lowered below a ‘critical temperature’ T_c , one observes an ordering of spins that lead to a

finite magnetization below T_c , rising continuously from its higher temperature zero value. One describes this as a situation where a symmetry is ‘spontaneously broken’. The symmetry with respect to the sign of the magnetization, broken in the actual state of the system, results in the emergence of different macroscopic states, related to each other by the symmetry so broken. Thus, below T_c , at zero field, two equivalent states of magnetization become possible, which are equivalent in all respects other than the sign of the magnetization, but a finite field will favour one of them over the other. Crossing the value $h = 0$ for $T < T_c$ will lead to a discontinuous change from one state to the other, the equivalent in the Ising case of the discontinuous change from the liquid to gas states of water that we described above for water.

Although continuous, the change in the magnetization at the critical point is singular, and is described by the form,

$$M(T, h = 0) \sim (T_c - T)^\beta \sim t^\beta, \quad (10)$$

where $t \equiv |T_c - T|/T_c$ and \sim (here and throughout) represents the fact that we focus only on the ‘important’ or singular component of the variation, and do not place importance on pre-factors and regular variations that may also be present. The exponent β is our first example of a critical exponent, whose predicated mean field value is $1/2$, quite distinct from the most recent experimental and theoretical estimates of about 0.326 . While the magnetization exhibits continuous change, its derivative with respect to the applied field, and the derivative of the internal energy with respect to temperature, do not. Both the susceptibility, and the heat capacity become arbitrarily large at the critical point, diverging as a power of the distance from the critical point, defining more critical exponents. Alphabetically first, of course, is the exponent α , which describes the manner in which heat capacity diverges,

$$C(T, h = 0) \sim (T_c - T)^\alpha \sim t^\alpha. \quad (11)$$

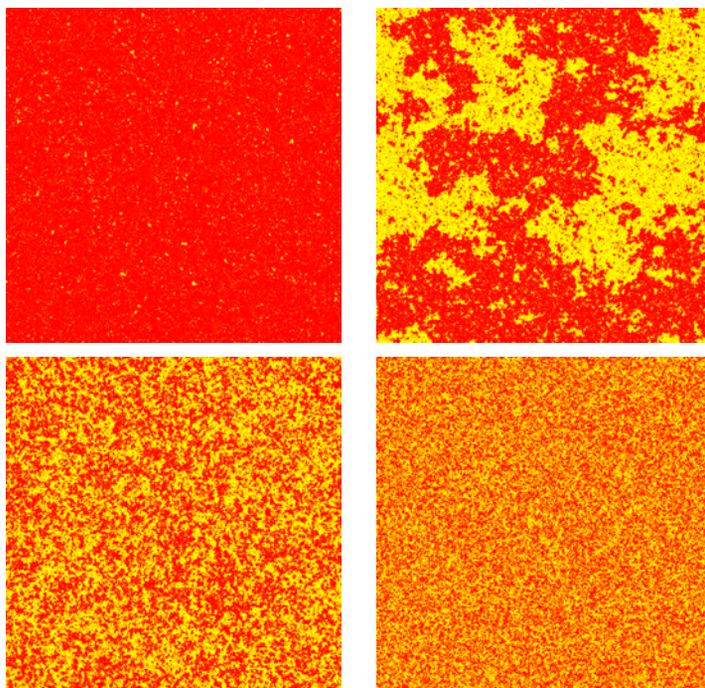
The susceptibility diverges as,

$$\chi(T, h = 0) \sim (T_c - T)^\gamma \sim t^\gamma. \quad (12)$$

The variation of the magnetization at the critical temperature, as the applied field is increased, defines the next critical exponent δ ,



Figure 6. Snapshots from Monte Carlo simulations of the Ising model in two dimensions at zero field (a) below T_c (b) at T_c (c) above T_c (d) a much higher temperature above T_c (from left to right, top to bottom). Red colour pixels represent +1 spins and blue -1 spins.



$$M(T_c, h) \sim h^{1/\delta} . \quad (13)$$

Considering the behaviour of the susceptibility, based on its definition in terms of the correlation function $G(r)$, one may ask what the divergence of the susceptibility may mean for the correlation function. Since χ is the integral of $G(r)$, we may conclude that as the critical point is approached, the correlation function does not go to zero fast enough for the integral of $G(r)$ to be a small number. Indeed, the closer the temperature is to T_c , the slower is the decay of $G(r)$, leading to a larger susceptibility. This can be seen visually in *Figure 6*. For temperatures above T_c , the ‘clustering’ of spins of the same sign grow as T_c is approached. One may think of this clustering in terms of a typical size scale of clustering, which are seen to grow as the critical point is approached. This typical size defines the *correlation length*, ξ , in terms of which the correlation function is given as,

$$G(r) = G_0 \frac{k_B T}{\gamma} r^{-(d-2+\eta)} \exp(-r/\xi), \quad (14)$$



which defines the exponent η .⁴

The correlation length grows arbitrarily large as the critical point is approached, and diverges at the critical point:

$$\xi \sim t^\nu. \tag{15}$$

At the critical point, therefore, the correlation function becomes ‘long ranged’, decaying only algebraically, as can be seen in *Figure 6* which displays (albeit within a finite window) correlated clusters of spins on all length scales. *Figure 6* also shows ordering of spins below T_c , whose clusters appear equally (or more) long ranged, but one must recall that in order to evaluate correlations, one must subtract the average value of the spins. In all but *Figure 6(a)*, the average value, or the magnetization, is zero, whereas in *Figure 6(a)*, it is finite. Indeed, below T_c , the correlation length is finite, even though clusters of spins span the entire system.

A large number of critical exponents, six, have been defined above, and a successful theoretical description of the critical point must provide an explanation of how they have the values they do, which are summarised in *Table 1*. It turns out, however, that these exponents are not all independent. A number of relations have been discovered between them, listed below,

$$\alpha + 2\beta + \gamma = 2, \tag{16}$$

$$\gamma = \beta(\delta - 1), \tag{17}$$

$$2 - \alpha = \nu d, \tag{18}$$

$$\frac{\gamma}{\nu} = 2 - \eta. \tag{19}$$

Together, these relations mean that only two out of the six critical exponents are independent. This is a simplification, but naturally begs the question of where the exponent relations above come from. These relations arise from some remarkable *scaling* relations between the various quantities. Through the work of Benjamin Widom and others, this led to the *static scaling hypothesis*, that the singular part of the free energy is a *generalized scaling homogeneous function*. What does it mean?

⁴We have stated this form here without explanation, but the general form can be derived easily enough, using, e. g., Landau theory, although these treatments predict $\eta = 0$.



Table 1. Exponent values are from (1) J V Sengers and J G Shanks, *J. Stat. Phys.*, Vol.137, 857, 2009, (2) M Campostrini, A Pelissetto, P Rossi and E Vicari, *Phys.Rev. E*, Vol. 65, 066127, 2002, up to three decimal places.

Exponent	Mean Field	Experiment Better Theory
α	0	0.11
β	1/2	0.326
γ	1	1.237
δ	3	4.79
ν	1/2	0.630
η	0	0.036

A function is said to be homogeneous if it satisfies,

$$g(x) = b^a g(bx). \tag{20}$$

A simple example is a function that has the form $g(x) = ax^c$. Then $g(bx) = b^c ax^c = b^c g(x)$ or $g(x) = b^{-c} g(bx)$.

An example in our context of such a function is the correlation function at the critical point. If we consider the correlation function $G(r)$ at T_c , we can write it as,

$$G(r) = b^{-(d-2+\eta)} G(b^{-1}r), \tag{21}$$

which we can easily verify from the form $G(r) \sim r^{-(d-2+\eta)}$ at T_c . In words, the statement is that the correlation function at a distance r is the same as the correlation function at a smaller distance $b^{-1}r$ (we assume b is greater than 1), when the latter is multiplied by a factor $b^{-(d-2+\eta)}$. Note that in this case, what we are doing is a *scale transformation* for the distances, so the above relation can be seen as a statement of *scale invariance* at the critical point, with correlation of spins on all length scales, which we already saw visually demonstrated.

However, if we consider a temperature away from T_c , the distance scale is set by the correlation length ξ , and thus, if we do a scale transformation, the scaled ξ will become effectively smaller. Then, we may expect that the correlation function to ‘look the same’ under a scale transformation if we also correspondingly scale the temperature (or the distance from the critical point, t) to a higher value, so that the correlation length is correspondingly smaller. This can be done by considering the two variable function $G(r, t)$, and scaling both r and t , but with different powers,

$$G(r, t) = b^{-(d-2+\eta)} G(b^{-1}r, b^{1/\nu}t). \tag{22}$$

Note that in this case, what we are doing is a *scale transformation* for the distances, so the above relation can be seen as a statement of *scale invariance* at the critical point, with correlation of spins on all length scales, which we already saw visually demonstrated.



This is an example of a generalized homogeneous function, which we define as one that has the property,⁵

$$g(x, y) = bg(b^a x, b^d y). \quad (23)$$

Choosing $b = t^{-\nu}$ (which is the same as scaling distances with the correlation length, since we have $\xi \sim t^{-\nu}$), we have,

$$G(r, t) = t^{\nu(d-2+\eta)}G(t^\nu r). \quad (24)$$

Since we obtain the susceptibility from integrating the correlation function, we can easily verify that the susceptibility exponent γ is given by,

$$\gamma = (2 - \eta)\nu, \quad (25)$$

which is one of the exponent relations written above, as $\gamma/\nu = (2 - \eta)$.

The scaling hypothesis for the free energy can be written in the general form,

$$f(t, h) = b^a f(b^{1/\nu} t, b^c h). \quad (26)$$

By choosing $b = t^{-\nu}$ as before, we can write,

$$f(t, h) = t^{-\nu a} f(h/t^\Delta), \quad (27)$$

where $\Delta = c\nu$. This illustrates that the independent parameters appear in specific combinations under the assumption of scaling. However, since the second derivative of f is the heat capacity that varies as $t^{-\alpha}$, we must have $-\nu a = 2 - \alpha$. Thus,

$$f(t, h) = t^{2-\alpha} f(h/t^\Delta). \quad (28)$$

Taking the first and second derivatives of f with respect to h to obtain the magnetization and susceptibility, and equating the result at $h = 0$ to the known exponents, we get,

$$2 - \alpha - \Delta = \beta, \quad (29)$$

$$2 - \alpha - 2\Delta = -\gamma. \quad (30)$$

⁵The different powers of b that multiply the right hand side should not cause confusion: The statement $g(x, y) = bg(b^a x, b^d y)$ can also be written as $g(x, y) = b^k g(b^{a'} x, b^{d'} y)$, where $a' = ak$ and $d' = dk$.



The scaling hypothesis provides some rationalisation of the exponent relations, but at this stage, it is a purely formal exercise, which does not provide us physical insight or an understanding of how such scaling behaviour arises. To address this, we next turn to the block spin transformation explored by Kadanoff.

Combining the two, we get,

$$\alpha + 2\beta + \gamma = 2, \tag{31}$$

which is another of the relations we have listed above. Other relations can be similarly obtained.

These relations have been studied experimentally and found to be valid. For example, consider the relation for the magnetization. From what we have said earlier about the free energy, we have,

$$M(T, H) = t^\beta M(h/t^\Delta). \tag{32}$$

It turns out that $\Delta = \gamma + \beta = \beta\delta$. Thus, if we plot $M(T, h)/t^\beta$ against $h/t^{\gamma+\delta}$ we should have the same behaviour regardless of what combination of h and t leads to a given value of $h/t^{\gamma+\delta}$.

An example is shown in *Figure 7*, which shows scaling. Note however that the scaling function is different for $T < T_c$ and $T > T_c$. Many such experimental results provide compelling evidence for the scaling hypothesis.

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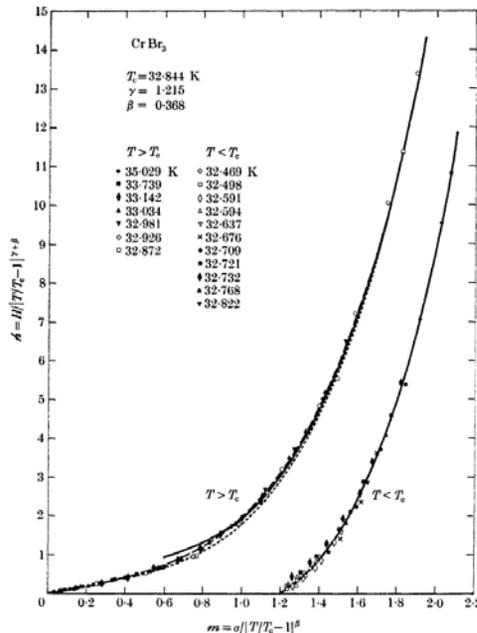


Figure 7. Scaling behavior of the magnetization. (From: J T Ho and J D Litster, *Phys. Rev Lett.*, Vol.22, 603, 1969)

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Kadanoff ‘Block Spin’ Transformation

We have discussed above the significant observation of universality in the critical behaviour of systems belonging to the same universality class. We have also indicated that correlations arise in these systems on larger and larger length scales as the critical point is reached. Thus, it seems reasonable to think that the ‘structure’ at large length scales is what is important in determining the behaviour near the critical point, and the right procedure to follow is to average over details on smaller, atomic, lengths, which might result in a universal description of critical behaviour, perhaps along with non-universal numbers that are different for different specific substances⁶. However, how to derive such a description needs to be developed. Further, given that heterogeneity and correlations persist at all length scales near the critical point, it is also not clear when such a procedure may be terminated. For any given temperature one may say that one must average, or ‘coarse grain’, over lengths up to the correlation length. However, this length scale itself changes very strongly as the critical point is approached.

Kadanoff undertook an analysis using the procedure of transforming the description of an Ising system, by defining ‘block spins’. The idea is that when we have a sufficiently large correlation length, one may choose blocks of spins of size such that (a) one expects all the spins within the block to have more or less the same sign, and (b) the block size is small compared to the correlation length. Then, the relevant degrees of freedom are not the individual spins but the block spins. Pursuing this line of thinking, a key insight put forward by Kadanoff was the idea that a block spin transformation is, in effect, a mapping of the system under study to a different system, characterised by different interaction strength and applied fields. In other words, a block spin transformation maps the system from one set of interaction pa-

The right procedure to follow is to average over details on smaller, atomic, lengths, which might result in a universal description of critical behaviour.

⁶A good analogy for the latter part may be the equations of fluid dynamics, which are used to describe different fluids, each fluid distinguished from the others by numbers like density and viscosity.



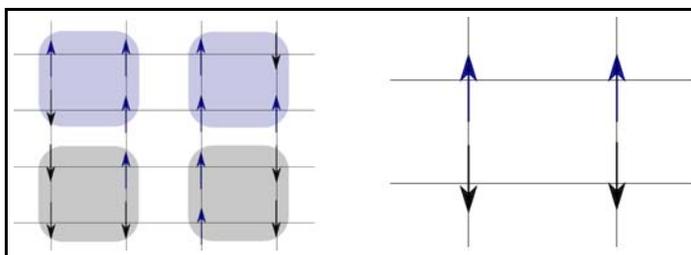
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rameters to another. The easiest way to see this is to consider what happens to the correlation length. Suppose we define a procedure wherein for each block of spins, we define a block spin based on a ‘majority rule’ – the block spin will have the sign of the majority of the spins (in case the \pm spins are the same in number, the spin value of a particular – pre-fixed – member of the block will be assigned to the block spin). As illustrated in *Figure 8*, this procedure will lead to a decrease in the number of spins, but at the same time an increase in the lattice spacing between the block spins. However, the lattice spacing is the natural unit with which to measure lengths, and in particular the correlation length. Clearly then, the result of the block transformation is a change in the correlation length to smaller values. Representing the coarse grained correlation length by $\tilde{\xi}$, and representing by b the block size, we can say $\tilde{\xi} = \xi/b$. Equivalently, keeping in mind the case of $T > T_c$ and $h = 0$, we can think of this as a rescaling of the temperature to a higher value by,

$$\tilde{t} = tb^{1/\nu}, \quad (33)$$

using the fact that $\xi \sim t^{-\nu}$. At temperatures below T_c , the effect is to map the system to temperatures that are lower, using the same logic. *Figure 9* illustrates the expected behaviour under successive coarse-graining steps. While the effective temperature will move away to either zero or infinite temperature with the starting temperatures below or above T_c , the critical point is not altered, reflecting the property of scale invariance. We say that the critical point is a fixed point under the block spin transformation. Behaviour under block spin transformations is illustrated in *Figure 10*, *Figure 11* and *Figure 12* for temperatures below, at and above the critical point.

Figure 8. Kadanoff block spin transformation. The illustration shows how a block of 2×2 spins is transformed into a single block spin.



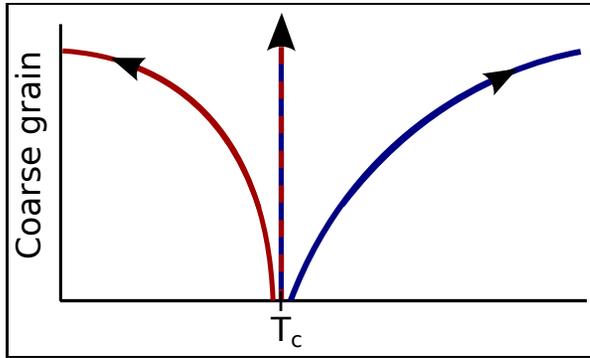


Figure 9. Change under coarse-graining

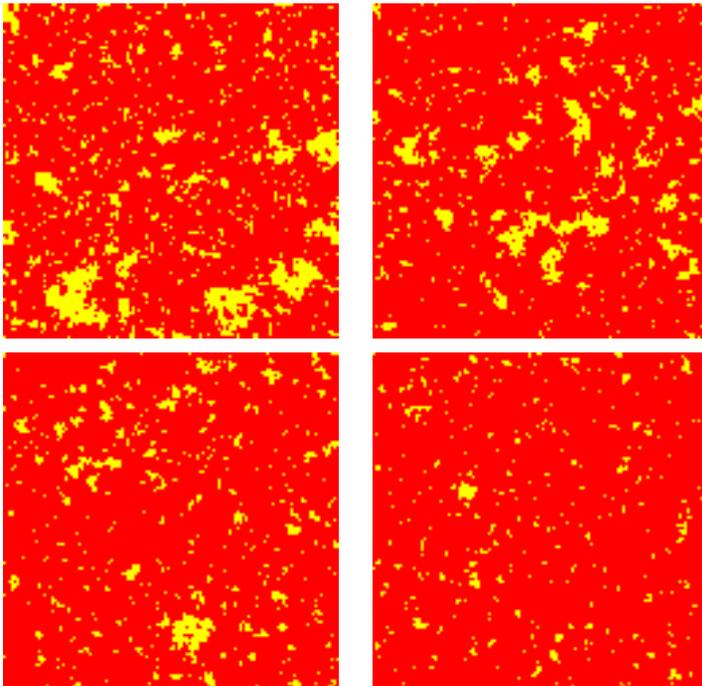


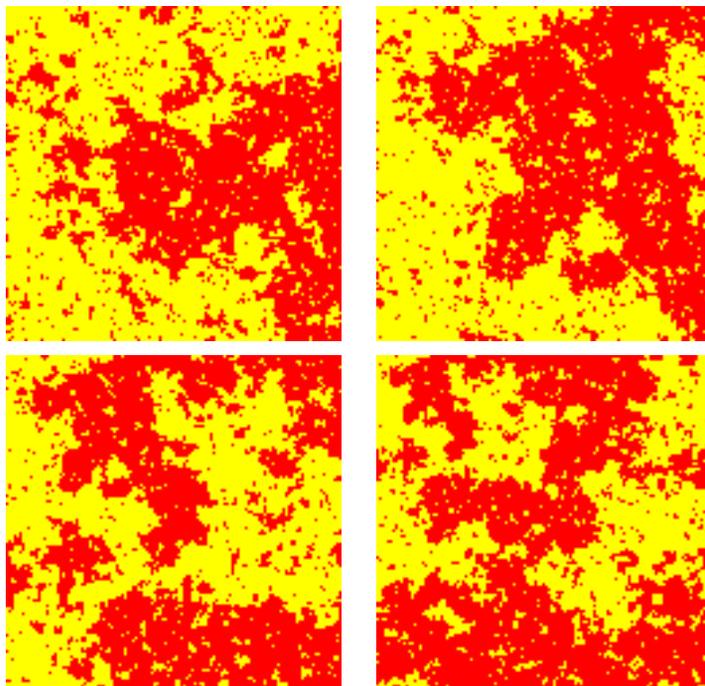
Figure 10. Snapshots from Monte Carlo simulations of the Ising model in two dimensions at zero field and temperature below T_c , $T = 2.24$, subjected to block spin transformation and rescaling. (a) no transformation (b) block size = 2 (c) block size = 4 (d) block size = 8. (from left to right, top to bottom). Red colour pixels represent +1 spins and blue -1 spins.

Kadanoff argued that the free energy of the original system, and that of the coarse grained system should be the same. Since the number of sites in the coarse grained system is smaller by b^d , we have $Nf(t, h) = (N/b^d)f(\tilde{t}, \tilde{h})$ or,

$$f(t, h) = b^{-d}g(\tilde{t}, \tilde{h}). \quad (34)$$



Figure 11. Snapshots from Monte Carlo simulations of the Ising model in two dimensions at zero field and temperature approximately at $T_c \sim 2.269185\dots$, $T = 2.27$, subjected to block spin transformation and rescaling. (a) no transformation, (b) block size = 2, (c) block size = 4 (d) block size = 8. (from left to right, top to bottom). Red colour pixels represent +1 spins and blue -1 spins.



Now, arguing that the applied field should also scale in a similar manner as the temperature, i. e., that $\tilde{h} = b^{\Delta/\nu}h$, we have,

$$f(t, h) = b^{-d} f(b^{1/\nu}t, b^{\Delta/\nu}h), \tag{35}$$

which is the scaling form that we saw earlier. Further, choosing $b = t^{-\nu}$, we get,

$$f(t, h) = t^{-\nu d} f(h/t^{\Delta}), \tag{36}$$

which is the same scaling form as before, with $\nu d = 2 - \alpha$ which is obtained as a prediction, if we go through the process of obtaining an expression for the heat capacity from the free energy, and using the definition that the corresponding critical exponent is α .

The Kadanoff construction contains tremendous physical insight, and results in providing a basis for the existence of scaling and the description of the free energy in terms of generalized homogeneous functions. However, it has some significant drawbacks. The first is that it does not provide a calculation scheme for evaluating the critical exponents. Second, it makes assumptions which



are in detail not correct, namely that the coarse-graining procedure will lead to a new description of the system in which the type of interactions between block spins will be the same as the original model. However, some thought will reveal that when we perform a block transformation, new interactions (such as those with next to nearest neighbours) may arise, if one were to insist that the block interaction energy captures the original interactions faithfully. These then have to be kept track of in providing a satisfactory description of the consequences of block transforms. These drawbacks are successfully addressed with the development of the renormalization group methods.

Renormalization

The coarse-graining procedure, followed by rescaling that we described above, is an example of what one refers to as a renormalization group (RG) transformation. The topic of renormalization deserves a considerably longer description than what we give here. The brief sketch here merely serves to reach a logical

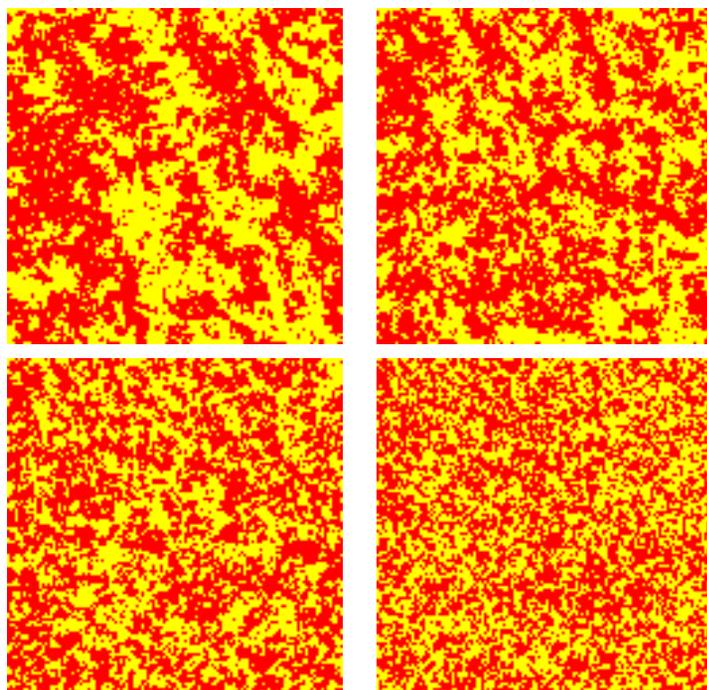


Figure 12. Snapshots from Monte Carlo simulations of the Ising model in two dimensions at zero field and temperature above T_c , $T = 2.5$, subjected to block spin transformation and rescaling. (a) no transformation (b) block size = 2 (c) block size = 4 (d) block size = 8 (from left to right, top to bottom). Red colour pixels represent +1 spins and blue -1 spins.



stopping point for the description of critical behaviour we have dealt with so far, and to point the direction in which analysis can proceed further.

Based on what we said about the Kadanoff block spin transformation, we shall now incorporate the possibility that a renormalization group transformation will in general lead to the generation of a whole range of new interaction terms. As a concrete example, we shall consider the Ising model, for which, following convention, we can write the Hamiltonian, scaled to the temperature as,

$$\frac{H}{k_B T} = K_1 \sum_i S_i + K_2 \sum_{\langle ij \rangle} S_i S_j. \quad (37)$$

Under a renormalization group transformation, we expect that that new couplings K will be generated, and in order to include them, we write a more general version,

$$\frac{\tilde{H}}{k_B T} = K_1 \sum_i S_i + K_2 \sum_{\langle ij \rangle} S_i S_j + K_3 \sum_{\langle ijk \rangle} S_i S_j S_k \dots \quad (38)$$

Now, if we consider the space that is defined by the set of all possible couplings, K_1, K_2 etc, define a vector \mathbf{K} . Under successive transformations, a point in this space, denoted by \mathbf{K} will transform, or *flow*, in some fashion. This can be represented by a mapping,

$$\mathbf{K}^{n+1} = R(\mathbf{K}^n). \quad (39)$$

From the discussion earlier, we expect that there will be special points, called fixed points, represented by the condition that these points will map onto themselves under a transformation,

$$\mathbf{K}^* = R(\mathbf{K}^*). \quad (40)$$

At such points, the correlation length will be either zero or infinity, and we are interested in the transformation behaviour near the latter type of fixed point. We can consider a linearized version of the mapping R near the fixed point, and write,

$$\mathbf{K}^{n+1} - \mathbf{K}^* = \mathbf{W}(\mathbf{K}^n - \mathbf{K}^*) \quad (41)$$

where \mathbf{W} is a matrix that represents the linearization of R . If one identifies the eigenvector directions of \mathbf{W} whose eigen values are



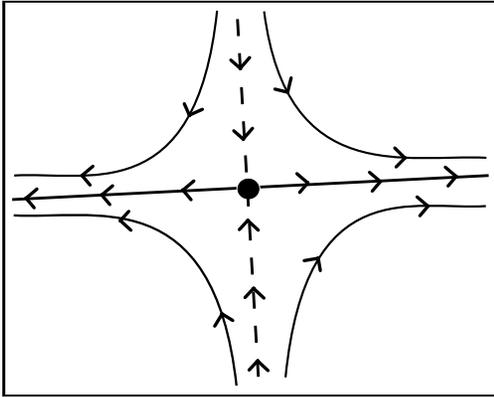


Figure 13. Schematic representation of flow under renormalization group transformations. The black dot in the middle represents the fixed point, and the horizontal axis represents a relevant direction (like, e.g., the temperature) along which the flow is away from the fixed point. Other flows, either converging to the fixed point or of mixed character, are also shown.

greater than 1 (the so called ‘relevant’ directions), one may write these eigenvalues as $\lambda = b^y$. The exponents y will correspond to the critical exponents, and the corresponding eigen directions will provide us information about the relevant fields for the system. A schematic of this scenario is given in *Figure 13*, which show the flow under RG transformations along relevant and irrelevant directions, and also in intermediate cases. This, in a nutshell, is the renormalization procedure that may be followed in order to elucidate the nature of critical points.

Summary

Salient features of continuous phase transitions, exemplified by the liquid-gas critical point and the critical point in magnetic systems, are the emergence of correlations that diverge as the critical point is approached, *universality* across many systems and phase transitions that belong to the same *universality class*, and the observation of interesting scaling behaviours and relationships near the critical point. Application of block transformations on the degrees of freedom, and studying how they transform the description of the system, leads to insights into critical behaviour, and the renormalization group procedure to elucidate and compute quantitative information about critical points, and into the manner in which universality emerges in classes of systems that exhibit the same critical behaviour.



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