The Embedding Theorems of Whitney and Nash

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We begin by briefly motivating the idea of a manifold and then discuss the embedding theorems of Whitney and Nash that allow us to view these objects inside appropriately large Euclidean spaces.

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1. Introduction

Let us begin by motivating the concept of a manifold. Start with the unit circle C in the plane given by

$$x^2 + y^2 = 1.$$

This is not a graph of a single function $y = f(x)$. However, apart from $(\pm 1, 0)$, C is the union of the graphs of $y = \pm\sqrt{1-x^2}$. Each graph is in one-to-one (bijective) correspondence with its domain, namely the interval $(-1, 1)$ on the x-axis – apart from the end points ± 1. To take care of points on C near $(\pm 1, 0)$, we see that apart from $(0, \pm 1)$, C is the union of the graphs of $x = \pm\sqrt{1-y^2}$. Again, each graph is in bijective correspondence with its domain, namely the interval from $(0, -1)$ to $(0, 1)$ on the y-axis. Thus, to accommodate the two exceptional points $(\pm 1, 0)$ we had to change our coordinate axes and hence the functions that identify the pieces of C. Doing all this allows us to describe *all* points on C in a bijective manner by associating them with points either on the x-axis or the y-axis. It is natural to ask whether such a bijection can be given for all points of the circle in a single stroke without having to resort to a two-step approach given above. The answer is no.

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Manifold, embedding, immersion, Nash, Whitney, smooth map, chart, Riemannian metric.

Let us look at another example. We consider the sphere S in $\mathbb{R}^3$ given by

$$x^2 + y^2 + z^2 = 1.$$

The theorems of Whitney and Nash concern general spaces called 'compact manifolds' like the circle and the sphere.

As before, it is possible to solve for one of the variables in terms of the other two. For example, writing $z = \pm\sqrt{1-x^2-y^2}$ exhibits two hemispheres of S as graphs over the unit disc in the xy-plane. The analog of the exceptional points in the case of C above is a much bigger set, namely the 'equator' $x^2+y^2=1$. To accommodate it, we write y (or x) in terms of the other two variables and proceed as before. Again, the conclusion of doing all this is that we are able to create a bijective correspondence between *all* points on S and discs that lie in the xy, yz or the zx planes. If we ask whether it is possible to build such a correspondence at one stroke instead of so many steps, the answer again is no. Both C and S furnish examples of what are called *manifolds*. These are spaces that 'locally' look Euclidean – that is, locally look like an open interval or an open rectangular region. Indeed, on C, consider the point $(0,1)$ and note that the entire upper arc is a neighbourhood containing it, that is in bijective correspondence with an interval $(-1,1)$ on the x-axis. The theorems of Whitney and Nash concern general spaces called 'compact manifolds' like the circle and the sphere; in a neighbourhood of each of its points, a manifold of dimension n looks like the n-space $\mathbb{R}^n$. The theorems address the question of whether these abstract spaces can be thought of as sitting inside $\mathbb{R}^N$ for some N. Of course, in doing so, there must be no distortions – the image must be an exact replica of the given manifold.

The purpose of this note is to provide an overview of some of the main ideas behind the embedding theorems of Whitney and Nash. To describe them briefly, we start with the definition of what we formally mean by a smooth real n-dimensional manifold M. The manifolds we define are firstly required to be well-behaved sets in the sense of topology; they are topological spaces that are second countable[1] and Hausdorff[2]. A homeomorphism between topological spaces is a one-to-one

[1] A topological space is second countable, if there exists some countable collection of open subsets such that any open subset of the space can be written as a union of elements of some subfamily of this countable collection. The property of being second countable restricts the number of open sets that a space can have. For the usual Euclidean space, the set of all open balls with rational radii and whose centers have rational coordinates, is countable and forms a basis as above.

[2] A Hausdorff space is a topological space in which any two different points admit disjoint open neighbourhoods. This property allows one to deduce uniqueness of limits of sequences. Hausdorff was one of the founders of topology.

continuous map whose inverse is also continuous. The notion of diffeomorphism is defined analogously when the continuity assumption is replaced by differentiability. A topological space M which is second countable and has the Hausdorff property, can be equipped with a so-called smooth structure which enables us to do differential calculus near each point. More precisely, a smooth structure on M consists of pairs (U_α, ϕ_α) where U_α is open in M and

$$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$$

is a homeomorphism such that M is covered by the union of the U_α's, and for all α, β with $U_\alpha \cap U_\beta \neq \emptyset$, the homeomorphism

$$\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta) \qquad (1)$$

between open sets in $\mathbb{R}^n$ is infinitely differentiable. Thus, M is made by gluing together open pieces (usually called 'charts') that are equivalent to open subsets of $\mathbb{R}^n$. Furthermore, the collection $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas; it is maximal in the sense that it contains all possible pairs (U_γ, ϕ_γ) that satisfy the compatibility condition (1). The pair (U_α, ϕ_α) can be thought of as giving coordinates since points in U_α can be assigned coordinates by identifying them with their images $\phi_\alpha(p) \in \mathbb{R}^n$. There is, of course, an ambiguity in this process since the same point on M can belong to several such coordinate charts. This is accounted for by viewing the transition maps $\phi_{\alpha\beta}$ as a change of coordinates. The manifold M is said to have dimension n.

Loosely speaking, Whitney's theorem – there are, in fact, several such – says that every smooth manifold can be viewed as sitting inside $\mathbb{R}^N$ for some large N, i.e., it can be *embedded* in $\mathbb{R}^N$. The word 'embedding' is used in a technically precise sense which will be explained shortly. It is of interest to determine the least N that works for all manifolds with a given dimension.

'Riemannian metric' provides a notion of length of a tangent vector and hence a way to measure the distance between two points.

Taking this one step further, we may consider a notion of distance on a manifold M; this is a so-called 'Riemannian metric' g on M. It provides a notion of length of a tangent vector and hence a way to measure the distance between two points. The pair (M, g) is called a 'Riemannian manifold'. More precisely, fix $p \in M$ and let $x = (x^1, x^2, \ldots, x^n)$ be local coordinates near p. For a tangent vector $v = (v^1, v^2, \ldots, v^n)$ in T_pM, the tangent space to M at p, the square of the length of v is given by

$$g(v, v) = \sum_{i,j=1}^{n} g_{ij}(x)v^i v^j \ ,$$

where the matrix $(g_{ij}(x))_{1 \leq i,j \leq n}$, each of whose entries is a smooth function of x, is symmetric and positive definite. Now, there is a natural notion of a submanifold of a manifold, and every submanifold S of $\mathbb{R}^n$ is clearly Riemannian since the standard inner product on $\mathbb{R}^n$ may be restricted to S. Nash's theorem says that not only can we embed M in $\mathbb{R}^N$ (for some N) but we can do this in such a way that the intrinsic notion of length of a tangent vector on M is inherited from $\mathbb{R}^N$. We discuss these theorems in more detail in Sections 2 and 3 wherein we adopt the convention that 'smooth' will always mean infinitely differentiable (denoted by C^∞), either for manifolds or maps between them.

2. The Whitney Embedding Theorem(s)

Let us begin by recalling two definitions. Let M, N be smooth manifolds of dimensions m, n respectively and $f : M \to N$ a smooth map. The tangent space at a point $p \in M$ is a linear space $\mathbb{R}^n$; it is the space of all directions in which one can pass through p tangentially. The above informal description is dependent on thinking of the manifold as sitting inside a Euclidean space already. However, there is an intrinsic definition of tangent vectors and tangent space at each point which we do not recall here. The key property of tangent spaces is that a smooth map $f : M \to N$ gives a natural *linear* map

$$df(p) : T_pM \to T_{f(p)}N.$$

The map f is said to be an *immersion* if the derivative map $df(p) : T_pM \to T_{f(p)}N$ is injective (another name for one-to-one maps) for all $p \in M$, and f is called a *submersion* if $dfp)$ is surjective (another name for onto maps) for all $p \in M$.

The map f is said to be an *immersion* if the derivative map $df(p) : T_pM \to T_{f(p)}N$ is injective.

A stronger notion is that of embedding. We say that $f : M \to N$ is an *embedding* provided f is an immersion which maps M homeomorphically onto its image $f(M) \subset N$. In other words, when $f(M)$ is equipped with the subspace topology from N, the spaces M and $f(M)$ are homeomorphic.

Examples of immersions that are not embeddings abound. For example, the derivative of the map $\psi : \mathbb{R} \to \mathbb{C}$ given by $\psi(t) = (\cos t, \sin t)$ is $d\psi(t) = (-\sin t, \cos t)$ which is injective for all t (since both components cannot vanish simultaneously), but ψ fails to be an embedding since it is not injective.

More examples can be constructed by considering the parametrizations of curves in $\mathbb{R}^2$ that have self-intersections; a much-quoted one being the map $\varrho : [0, 2\pi] \to \mathbb{R}^2$, $\varrho(t) = (\sin 2t, -\sin t)$. This traces a standing figure-of-eight as t increases from 0 to 2π. It can be checked that ϱ restricted to $(0, 2\pi)$ is an injective immersion that fails to be an embedding. A useful observation is that if M is compact, then any injective immersion $f : M \to N$ is an embedding. With all this out of the way, here is then the first statement in a hierarchy of embedding theorems due to Whitney.

We say that $f : M \to N$ is an *embedding* provided f is an immersion which maps M homeomorphically onto its image $f(M) \subset N$.

Theorem 1. *Any compact manifold M of dimension n can be embedded in $\mathbb{R}^N$ for sufficiently large N.*

Proof. Let $B^n(q, r)$ denote the open ball in $\mathbb{R}^n$ centered at $q \in \mathbb{R}^n$ with radius $r > 0$. Since M is compact, there are finitely many coordinate charts (U_α, ϕ_α), $1 \leqslant \alpha \leqslant k$, such that $\phi_\alpha(U_\alpha) \supset B^n(0,2)$ and M is covered by the union of the finitely many open sets $V_\alpha = \phi_\alpha^{-1}(B^n(0,1))$. Let $\varrho_\alpha : M \to \mathbb{R}^n$ be smooth functions supported in U_α such that $\varrho_\alpha \equiv 1$ in a neighbourhood of $\overline{V}_\alpha$. Thus $\psi_\alpha(x) = \varrho_\alpha(x)\phi_\alpha(x)$ is a well-defined smooth map on M that vanishes outside U_α. Also, $\psi_\alpha = \phi_\alpha$ on V_α. Define $f : M \to \mathbb{R}^{k(n+1)}$ by

$$f(x) = (\psi_1(x), \psi_2(x), \ldots, \psi_k(x), \varrho_1(x), \varrho_2(x), \ldots, \varrho_k(x)).$$

Note that f is smooth. To see that f is injective, suppose that $f(x) = f(y)$. Then $\psi_\alpha(x) = \psi_\alpha(y)$ and $\varrho_\alpha(x) = \varrho_\alpha(y)$ for all $\alpha \leqslant k$. Also, there exists at least one index, say α_0, for which $\varrho_{\alpha_0}(x) = \varrho_{\alpha_0}(y) \neq 0$ since the V_α's cover M and $\varrho_\alpha \equiv 1$ on V_α. Thus, $\psi_{\alpha_0}(x) = \psi_{\alpha_0}(y)$ implies that $\phi_{\alpha_0}(x) = \phi_{\alpha_0}(y)$. This in turn gives $x = y$ as ϕ_{α_0} is bijective. Incidentally, this shows that both x, y must be in V_{α_0}.

To check that f is an immersion, we must inspect the derivative df. Fix an arbitrary $p \in M$ and suppose that $p \in V_\beta$ for some $\beta \leq k$. The rows of the matrix of

$$df(p) : T_pM \to T_{f(p)}\mathbb{R}^{k(n+1)} \simeq \mathbb{R}^{k(n+1)}$$

are created by taking the derivatives of the various components of f. In other words, we may decompose the matrix of $df(p)$ into blocks, each of which represents the derivative of the various components of f. Near p, $\varrho_\beta \equiv 1$ and hence $\psi_\beta = \phi_\beta$. This means that near p, the map f looks like

$$f(x) = (\psi_1(x), \psi_2(x), \ldots, \phi_\beta(x), \ldots, \psi_k(x), \varrho_1(x), \varrho_2(x), \ldots, 1, \ldots, \varrho_k(x))\,.$$

Since ϕ_β is a smooth diffeomorphism near p, its derivative has rank n as a linear map at p. The derivative of ϕ_β is also present in $df(p)$ as a sub-block. Hence $df(p)$

has full rank, i.e., n at p. Since M is compact, f being an immersion must be an embedding. This completes the proof. $\square$

A stronger theorem due to Whitney is the following one.

Theorem 2. *Any compact manifold M of dimension n can be embedded in $\mathbb{R}^{2n+1}$ and immersed in $\mathbb{R}^{2n}$.*

Proof. By the above theorem, M embeds in some $\mathbb{R}^N$, where we may assume that $N > 2n+1$ without loss of generality. As a result, we may replace M by its embedded image and think of it as a submanifold of $\mathbb{R}^N$. Then suppose that $M \subset \mathbb{R}^N$ where $N > 2n+1$. Whitney showed that there exists a linear map from $\mathbb{R}^N$ to a suitable copy of $\mathbb{R}^{N-1}$ which restricts to the identity function on this copy of $\mathbb{R}^{N-1}$ (that is, the map is a projection from $\mathbb{R}^N$ to this $(N-1)$ dimensional subspace), and whose restriction to M is again an embedding provided $N > 2n+1$. By repeating this argument, it will follow that M eventually embeds in $\mathbb{R}^{2n+1}$. To get a hold of this linear projection, let $v \in \mathbb{R}^N$ be a non-zero vector based at the origin and consider the orthogonal complement

$$W_v = \{u \in \mathbb{R}^N : \langle u, v \rangle = 0\} \simeq \mathbb{R}^{N-1}.$$

Here, the angle brackets denote the standard inner product on $\mathbb{R}^N$. Let $P_v : \mathbb{R}^N \to W_v$ be the orthogonal projection, which, as may be recalled, is given by

$$P_v(x) = x - \frac{\langle x, v \rangle}{\langle v, v \rangle} v$$

for $x \in \mathbb{R}^N$. The main idea is to show that the set of v's for which $P_v : M \to \mathbb{R}^{N-1}$ is not an embedding has measure zero. Note that P_v restricted to M fails to be an embedding when it is either not injective or not an immersion.

First, let us identify all those v's for which P_v is not injective on M. Suppose there are points $p_1 \neq p_2$ on M such that $P_v(p_1) = P_v(p_2)$. By using the definition of P_v above, this implies that

$$p_1 - p_2 = cv, \quad \text{where } c = \frac{\langle p_1, v\rangle}{\langle v, v\rangle} - \frac{\langle p_2, v\rangle}{\langle v, v\rangle}.$$

This means that $p_1 - p_2$ is parallel to v. It is convenient to look at homogeneous coordinates – that is, for a non-zero element $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, one considers the whole line through the origin and this point as a single point in the *projective space* $\mathbb{RP}^{N-1}$. One writes this point as $[x]$. So, by passing to homogeneous coordinates, the above statement says that $[p_1 - p_2] = [v] \in \mathbb{RP}^{N-1}$. Consider the map $\sigma_1 : (M \times M)\backslash\Delta \to \mathbb{RP}^{N-1}$ defined on the complement of the diagonal $\Delta = \{(m, m) : m \in M\}$ by

$$\sigma_1(p_1, p_2) = [p_1 - p_2]$$

where $\Delta = \{(p, p) : p \in M\}$ is the diagonal in M. The set of all v's for which P_v is not injective on M lies in the image of σ_1. Now $(M \times M) \setminus \Delta$ is open in $M \times M$ and hence has dimension $2n$, while $\mathbb{RP}^{N-1}$ has dimension $N - 1$. Since $2n < N - 1$ by assumption, the famous theorem of Sard[3] implies that the image of σ_1 has measure zero in $\mathbb{RP}^{N-1}$.

[3] Sard's lemma or Sard's theorem asserts that for a smooth map from a manifold to another, the set of critical values (that is, the image of the set of critical points) has Lebesgue measure zero. That is, the set of critical values is small.

Next, let us identify all those v's for which P_v fails to be an immersion on M. Suppose there is some $p \in M$ and a non-zero vector $X_p \in T_pM$ such that $dP_v(p)(X_p) = 0$. As P_v is linear, $dP_v = P_v$. Thus, $P_v(X_p) = 0$, i.e., X_p belongs to the kernel of P_v which is one-dimensional and spanned by v. In other words, $[X_p] = [v]$. Consider the map $\sigma_2 : TM \setminus \{0\} \to \mathbb{RP}^{N-1}$ defined by

$$\sigma_2(p, X_p) = [X_p].$$

Here, TM is the so-called tangent bundle of M consisting, by definition, of all pairs (p, X_p) with a point p in M

and a tangent vector X_p at p. Also, $TM \setminus \{0\}$ denotes $\{(p, X_p) : X_p \neq 0\}$. The set of all v's for which P_v is not an immersion on M is contained in the image of σ_2. The end game is a dimension count as above; indeed, the dimension of $TM \setminus \{0\}$ is $2n$ since it is open in TM (and which has dimension $2n$) while that of $\mathbb{RP}^{N-1}$ is $N-1$. Since $2n < N-1$ by assumption, Sard's theorem again implies that the image of σ_2 has measure zero in $\mathbb{RP}^{N-1}$. Thus, if we avoid the image of σ_1 and σ_2, both of which have measure zero, there is plenty of room to pick a v for which P_v is an injective immersion and hence an embedding on M.

To see that M can be immersed in $\mathbb{R}^{2n}$, we first get an embedding of M in $\mathbb{R}^{2n+1}$ as above. Consider the collection of all unit vectors in TM, i.e.,

$$TM_1 = \{(p, X_p) : |X_p| = 1\} \subset TM \setminus \{0\}$$

which is a $(2n-1)$-dimensional manifold. Restrict σ_2 to TM_1 to get

$$\sigma_2 : TM_1 \to \mathbb{RP}^{2n}.$$

By Sard's theorem, the image $\sigma_2(TM_1) \subset \mathbb{RP}^{2n}$ has measure zero. Thus, there is a vector v for which P_v is an immersion on M.

□

These theorems can be strengthened in several pages. First of all, the assumption that M be compact can be dropped. Whitney also showed that if $n > 1$, every n-dimensional manifold M can be immersed in $\mathbb{R}^{2n-1}$ and if $n > 0$, then every n-dimensional manifold M can be embedded in $\mathbb{R}^{2n}$.

3. The Nash Embedding Theorem(s)

Let (M, g) be a Riemannian manifold of dimension n. The problem now is to construct a map

$$u : M \to \mathbb{R}^N, \ u = (u_1, u_2, \ldots, u_N)$$

for some N so that the following conditions are satisfied. First, u must be an embedding, and second,

$$g = (du_1)^2 + (du_2)^2 + \ldots + (du_N)^2. \tag{2}$$

This says that g agrees with what one obtains by using the map u to 'pull back' the standard metric on $\mathbb{R}^N$. In other words, u serves as an isometric (that is, distance-preserving) embedding of M. Finding such a u is the isometric embedding problem. To motivate the main ideas here, let us write the metric g in local coordinates $x = (x_1, x_2, \ldots, x_n)$ as

$$g = \sum_{i,j=1}^{n} g_{ij}\; dx_i\; dx_j\, .$$

Now compute the derivatives du_i and compare coefficients of $dx_i\; dx_j$ in (2) to get

$$g_{ij} = \sum_{l=1}^{N} \partial_i u_l\; \partial_j u_l = \langle \partial_i u, \partial_j u \rangle, \tag{3}$$

where $\partial_k = \partial/\partial x_k$ is the partial derivative with respect to x_k. Since $1 \leqslant i, j \leqslant n$ and $g_{ij} = g_{ji}$, there are exactly $n(n+1)/2$ equations in (3). Solving (3) for u would give us a local isometric embedding. There are several steps in solving just the local problem, the first being to rewrite the metric g in a more convenient form. It turns out that for a given point $p \in M$, there is a local coordinate system $x = (x_1, x_2, \ldots, x_n) = (x', x_n)$ near p in which g can be written as

$$g = \sum_{i,j=1}^{n-1} g_{ij}(x', x_n)\; dx_i\; dx_j + dx_n^2.$$

Thus, the x_n-direction can be decoupled from the others. In particular, this helps in reducing the complexity of (3). Indeed, (3) is now equivalent to

$$\langle \partial_k u, \partial_n u \rangle = 0,\; \langle \partial_n u, \partial_n u \rangle = 1 \text{ and } \langle \partial_k u, \partial_l u \rangle = g_{kl}\, .$$

for all $1 \leqslant k, l \leqslant n-1$. By differentiating, they can be shown to be equivalent to yet another system of equations that involve $\partial_{nn}u, \partial_k u$ and $\partial_l u$ for $1 \leqslant k, l \leqslant n-1$. The next step is this: If $\partial_k u, \partial_{kl}u, \partial_n u$, $1 \leqslant k, l \leqslant n-1$ are linearly independent near p, then locally this system of equations can be solved for $\partial_{nn}u$ in terms of the other partial derivatives of u, i.e.,

$$\partial_{nn}u = F(x, \partial_k u, \partial_n u, \partial_{kl}u, \partial_{kn}u), \qquad (4)$$

where F is smooth in x and real-analytic in the other arguments. If the metric g is real-analytic, then so is F in the variable x. Thus, for real-analytic metrics g, (4) can be solved by appealing to the Cauchy–Kowalevski theorem[4]. Note that this is valid if the condition on linear independence holds. But arranging this to hold is not difficult provided we are willing to increase N. Hence, there always exist local isometric embeddings for real-analytic metrics.

[4] This is the main theorem showing the local existence and uniqueness results for the Cauchy initial value problem for a system of nonlinear partial differential equations where the coefficients are analytic functions. There are examples showing that this is not always valid for smooth functions.

Several natural and interesting questions arise, like the problem of finding a global solution even for real-analytic metrics, the question of dealing with smooth metrics and finally, finding the optimal N that works for a given n. But, all these questions require the full force of various techniques from the theory of partial differential equations which we cannot hope to touch upon here. We refer the interested reader to look at [1] for a detailed presentation. It must be mentioned that Nash's original arguments [2] used a set of different ideas that were simplified to some extent by Moser [3] – both involved working with a suitable version of the implicit function theorem. When the metric g is real-analytic, it was shown by Greene and Jacobowitz [4] that the embedding can be chosen to be real-analytic as well, using methods of complex analysis.

There always exist local isometric embeddings for real-analytic metrics.

Suggested Reading

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