In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Pythagorean Theorem From Heron's Formula: Another Proof

Bikash Chakraborty
Department of Mathematics
University of Kalyani
West Bengal 741235, India
Email:
bikashchakraborty.math@yahoo.com

In this article, we give an alternative proof of Pythagorean theorem from Heron's formula using elementary school-level geometry.

For a right-angled triangle ABC (where ∠BAC is the right angle), we have to show that

$$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2.$$

For the proof, let O be the mid point of BC. Since ∠BAC is the right angle, we can draw a circle with center O and radius $\overline{BO}$ such that the points A,B,C lie on the circle. Now we join the two points A and O and draw a line segment from B parallel to AC; this meets the circle at D, (say). Again we join the points C,D and O,D.

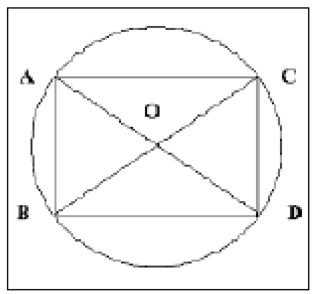

Figure 1.

Assume $\overline{AB} = a,\ \overline{AC} = b,\ \overline{BO} = r.$

Clearly, from *Figure* 1,

$$\begin{aligned}&\text{Area of the rectangle ABDC}\\ &\quad = 2(\text{area of the} \triangle \text{ ABO} + \text{area of the } \triangle \text{ AOC}).\end{aligned} \tag{1}$$

Keywords
Heron's formula, area of triangle, Pythagorean theorem.

Area of the $\triangle$ ABO

$$= \left|\sqrt{s(s-a)(s-r)(s-r)}\right|$$

$$= \left|\frac{a}{2}\sqrt{r^2 - \frac{a^2}{4}}\right|,$$

where $s = \frac{a+r+r}{2}$, and $|x|$ denotes the absolute value of x.

Similarly,

Area of the $\triangle$AOC

$$= \left|\frac{b}{2}\sqrt{r^2 - \frac{b^2}{4}}\right|.$$

Thus from (1),

$$ab = \left|a\sqrt{r^2 - \frac{a^2}{4}}\right| + \left|b\sqrt{r^2 - \frac{b^2}{4}}\right|.$$

By putting $t = 4r^2 - a^2 - b^2$ in the above equation, we get

$$2ab = \left|a\sqrt{t+b^2}\right| + \left|b\sqrt{t+a^2}\right|, \qquad (2)$$

i.e.,

$$2a^2b^2 - (a^2+b^2)t = 2ab\left|\sqrt{t^2 + (a^2+b^2)t + a^2b^2}\right| .$$

Again squaring both sides, we get

$$(a^2-b^2)^2t^2 - 8a^2b^2(a^2+b^2)t = 0.$$

Now we consider two cases :

Case 1. $a = b$. Then, clearly $t = 0$, i.e., $4r^2 = a^2 + b^2$.

Case 2. $a \neq b$
In this case we have $t \geq 0$.
Now when $t > 0$, then

$$|a\sqrt{t+b^2}| + |b\sqrt{t+a^2}| > 2ab,$$

which contradicts (2).

So $t = 0$, i.e., $4r^2 = a^2 + b^2$.

Hence $\overline{AB}^2 + \overline{AC}^2 = a^2 + b^2 = (2r)^2 = \overline{BC}^2.$ Hence the proof.

References

[1] C Alsina, R B Nelsen, ***Icons of Mathematics***, **The Mathematical Association of America, Washington, DC, 2011.**

[2] **W Dunham, *Journey through Genius*, Penguin Books, 1991.**