Products of Sets: Ordered and Unordered

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In this article we study the (Cartesian) Products of sets. As a prelude to it, we examine various definitions offered by different mathematicians to the concept of an ordered pair. The subtle difference between the notions of ordered products of sets and unordered products of sets is highlighted.

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In the same vein, the author believes that, it is unsafe to take yourself for more of a fool than you are.**

1. Introduction

It is well known that the axiomatic method of mathematics started with the Greek geometers before 300 BC It is to the everlasting credit of these ancient Greeks that they realized that the edifice of mathematics needs to be founded upon a few 'axioms' or self-evident truths. The truth of these axioms is accepted without question. Of course, the axioms should not contradict each other. Since axioms demand an element of our faith in their truth, it is wise to limit the number of axioms. Just as the axioms are given the status of 'primitive' truths, there is a related notion of 'undefined terms'[1]. Just as there is a need to limit the number of axioms, there is a need to limit the number of undefined terms.

[1] A definition describes the meaning of a word in terms of other words. The vocabulary of any language is finite and so the activity of defining terms cannot be concluded without committing the sin of circularity (= defining a term in terms of itself). Hence we need to accept some words as understood without defining them. Such words are called undefined terms.

In elementary set theory, we are familiar with notions of relations and functions as sets of ordered pairs. In the early days of set theory, it was usual for an ordered pair to be considered an undefined term. Later, it was discovered by people like Norbert Wiener[2], Felix Hausdorff and Kazimierz Kuratowski (*Box* 1 that a definition of 'ordered pair' could be devised using the notion of a set. Thus there was no need to leave the notion of 'ordered pair' undefined. As already remarked, a reduction in the number of undefined terms is to be welcomed.

Keywords
Cartesian product, ordered pair, ordered product, unordered prduct.

Box 1.

Norbert Wiener (1894–1964) was a child prodigy who joined Harvard University for graduate studies in 1909 at the age of 14! He received his PhD from Harvard when he was 18 years old! He made phenomenal contributions to the theory of probability, communication theory, cybernetics, harmonic analysis and many other mathematical subjects, (see *Resonance*, Vol.4, No.1, 1999).

Felix Hausdorff (1868–1942)was born into a Jewish German family. He was a man of broad interests. His interests included music, literature, philosophy, physics and mathematics. He also wrote poetry and many literary and philosophical books. He was awarded a PhD at the age of 23 for his work in the applications of mathematics to astronomy. He also worked on the applications of mathematics to optics. He made great contributions to set theory, topology, measure theory, probability and functional analysis. He committed suicide in 1942 since he was disgusted with the atrocities of the Nazi regime.

Kazimierz Kuratowski (1896–1980) was a Polish mathematician and logician. He is celebrated for his fundamental contributions to topology and measure theory. His axioms for a topological space via the closure operator are mentioned in almost every book on topology.

The desire to do so by inventing proper definitions for hitherto undefined terms has motivated a number of mathematicians to devise set-theoretic definitions for the concept of 'ordered pair'.

2. Ordered Pairs

Almost every book on higher mathematics has something or the other to do with products of sets. All such books that the author has come across consider ordered cartesian products of finitely many sets first and then proceed to unordered cartesian products of arbitrary families of sets without commenting on the subtle difference between the two kinds of products. This article intends to rectify this situation.

2.1 *Wiener's Definition of an Ordered Pair*

Our story begins with Norbert Wiener. In 1914, Wiener gave the first simple definition of an ordered pair $(x, y) = \{\{\{x\}, \emptyset\}, \{\{y\}\}\}$. To show that this is a good enough definition, we need to check that $(x, y) = (a, b)$ iff $x = a$ and $y = b$. Suppose $(x, y) = (a, b)$. (x, y) has a unique two-element set as one of its two members, namely, $P = \{\{x\}, \emptyset\}$. Likewise, (a, b) has $Q = \{\{a\}, \emptyset\}$ as its unique two-element member. For equality of (x, y) and (a, b) we must have $P = Q$ which easily implies $a = x$. Equating the unique one-element members of (a, b) and (x, y) we obtain $y = b$. To prove the converse, that $x = a$ and $y = b$ implies $(x, y) = (a, b)$ is a trivial task.

We shall now have a look at Felix Hausdorff's definition of an ordered pair. Before that let us paint a brief biographical sketch of Hausdorff.

2.2 *Hausdorff's Definition of an Ordered Pair*

We shall now have a look at Felix Hausdorff's definition of an ordered pair. Just a little while after Wiener gave his definition of an ordered pair, Hausdorff gave the definition $(x, y) = \{\{x, \emptyset\}, \{y, \{\emptyset\}\}\}$. Suppose $(x, y) = (a, b)$. We claim that $x = a,\ y = b$. Suppose, to obtain a contradiction, that $x \neq a$. Now $\{x, \emptyset\} \in (x, y) = (a, b)$ implies $\{x, \emptyset\} = \{a, \emptyset\}$ or $\{x, \emptyset\} = \{b, \{\emptyset\}\}$. If $\{x, \emptyset\} = \{a, \emptyset\}$ then $x = \emptyset$ since we have supposed that $x \neq a$. Hence the above equation of sets becomes $\{\emptyset\} = \{a, \emptyset\}$ implying $a = \emptyset = x$, a contradiction to $a \neq x$. If on the other hand $\{x, \emptyset\} = \{b, \{\emptyset\}\}$, it follows that $\emptyset = b$ and $x = \{\emptyset\}$ since $\emptyset \neq \{\emptyset\}$. Also $\{a, \emptyset\} \in (x, y)$ and $\{a, \emptyset\} \neq \{x, \emptyset\}$ as already seen. Hence $\{a, \emptyset\} = \{y, \{\emptyset\}\}$ from which it follows that $y = \emptyset$ and $a = \emptyset$. Hence $a = x = \{\emptyset\}$, a contradiction to our assumption that $x \neq a$. Hence $x = a$. The reader is invited to derive a contradiction by supposing $y \neq b$. To show that $x = a$ and $y = b$ implies $(x, y) = (a, b)$ is trivial.

2.3 *Kuratowski's Definition of an Ordered Pair*

Kagimierz Kuratowski came up with the definition $(x, y) = \{\{x\}, \{x, y\}\}$. The reader can easily check that this definition serves its purpose. There are several interesting and deep logical and philosophical issues relating to the definition of the concept of ordered pair and the issue is not yet settled! The interested reader may refer to [1],[2], or [3]. We strongly recommend [2].

3. Relations and Functions

Whatever definition we may adopt, the crucial fact about an ordered pair is that $(a, b) = (c, d)$ iff $a = c$ and $b = d$. Once the definition of an ordered pair is accomplished, it is easy to define the concepts of 'relation' and 'function'. If A, B are sets then the ordered cartesian product $A \times B$ of A and B in that order is defined as $A \times B = \{(a, b)/a \in A, b \in B\}$. Any subset R of $A \times B$ is called a relation from A into B. We sometimes write aRb to express that (a, b) is an element of the relation R. A relation from A into B is called function if $R(a)$ is a single element set for each a in A, where $R(a)$ is defined as the set $\{b/b \in B \text{ and } (a, b) \in R\}$. These facts are familiar even to high school students.

4. Unordered Product of Two Sets

Suppose P_{I} denotes the set of members of the Indian Parliament and P_{B} denotes its British counterpart. Suppose further that we are asked to find the ordered cartesian product of these sets. We cannot proceed to answer this question unless the questioner further informs us about the *order* in which the sets are to be taken for forming the product. However, defining and forming an unordered product of the sets is easy as shown below.

Let C be the set {India, Britain} and let $\prod\{P_{\mathrm{I}}, P_{\mathrm{B}}\}$ denote the set of all functions $f : C \to P_{\mathrm{I}} \cup P_{\mathrm{B}}$ satisfying $f(\text{India}) \in P_{\mathrm{I}}$ and $f(\text{Britain}) \in P_{\mathrm{B}}$. Each function f in $\prod\{P_{\mathrm{I}}, P_{\mathrm{B}}\}$ picks one Indian MP as the image of India and one British MP as the image of Britain. Thus f serves as an unordered pair of an Indian MP and a British MP. This process of forming an unordered product of sets can be easily extended to any number of sets, finite or infinite, as shown below.

4.1 *Unordered Cartesian Products of Arbitrary Family of Sets and Unordered* Δ*-tuples*

Suppose $\{A_\alpha/\alpha \in \Delta\}$ is a family of sets. We define the unordered Cartesian product $\prod\{A_\alpha/\alpha \in \Delta\} = \prod\limits_{\alpha\in\Delta} A_\alpha$ as the set of all functions $f : \Delta \to \bigcup\limits_{\alpha\in\Delta} A_\alpha$ satisfying $f(\alpha) \in A_\alpha$ for each α in Δ. If $B_\alpha = \{x_\alpha\}$ with $x_\alpha \in A_\alpha$ for each $\alpha \in \Delta$ then the unique element of $\prod\{B_\alpha/\alpha \in \Delta\}$ is called an unordered Δ-tuple.

5. Ordered Cartesian Products of Arbitrary Families of Sets

Having successfully defined unordered products of sets we now proceed to define ordered products. Even in the case of the ordered product of two sets A_1 and A_2we should specify whether A_1 sits in the first place to form $A_1 \times A_2$ or in the second place to form the product $A_2 \times A_1$. The first product has been formed by assigning precedence to index 1 over index 2, whereas the second is formed by assigning precedence to index 2 over index 1.

The proper mathematical setting for precedence relations is the theory of partially ordered sets or 'posets' for short. A relation $\leq$ on a nonempty set A is called a partial order or partial ordering on A if (i) $a \leq a$ (reflexivity), (ii) $a = b$ whenever $a \leq b$ and $b \leq a$ (antisymmetry) and (iii) $a \leq c$ whenever $a \leq b$ and $b \leq c$ (transitivity) for all a, b, c in A. If $\leq$ is a partial order on A then we say that $(A, \leq)$ is a poset. Observe that the three conditions of a partial order appear to capture the essential properties of a precedence relation. But appearances are deceptive! To make our point we appeal to an example. Let $A = \{1, 2, 3, 4, 5, 6\}$. Define $a \leq b$ if a divides b for a, b in A. Thus $\leq = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6),$

$(3,3), (3,6), (4,4), (5,5), (6,6)\}$. In this poset, 4 neither precedes 6 nor follows it. We need to impose one more condition on a poset to make it useful for our present purpose. A poset $(A, \leq)$ is called a totally ordered set or a chain if for each a, b in A either $a \leq b$ or $b \leq a$. For example, the set of real numbers is a chain under the usual ordering.

Suppose now $\mathfrak{A} = \{A_\alpha / \alpha \in \Delta\}$ is a family of sets. As already remarked, to form ordered products of the sets in $\mathfrak{A}$, the index set Δ needs to be totally ordered. So assume that $\leq$ is a partial order on Δ such that $(\Delta, \leq)$ is a chain. Just as $A_2 \times A_1$ is defined as the set of all ordered pairs with the first component from A_2 and the second from A_1 and just as $A_2 \times A_3 \times A_1$ is the set of all ordered triples with the first component from A_2, the second from A_3 and the third from A_1, we shall try to define Δ-tuples whose α-th component belongs to A_α and whose components follow the same ordering as the ordering $\leq$ on Δ. Our ordered product of the family of sets $\mathfrak{A}$ with ordering as in Δ will henceforth be denoted by $\prod(\mathfrak{A}, \leq)$. As already remarked, the definition of $\prod(\mathfrak{A}, \leq)$ will be complete if we describe its elements properly. So let us now take up the task of defining Δ-tuples. Broadly, each A_α in $\mathfrak{A}$ contributes one of its elements a_α to the Δ-tuple, and if α, β are elements of Δ with $\alpha \leq \beta$ then a_α precedes a_β. Thus each Δ-tuple is a set (with one element from each of the sets A_α, $\alpha \in \Delta$) with a partial order that is naturally inherited from the partial order available in Δ. The above discussion leads to the following formal definition of a Δ-tuple:

(i) T is the range of a function $f : \Delta \to \bigcup_{\alpha \in \Delta} A_\alpha$ satisfying $f(\alpha) \in A_\alpha$ for each α in Δ. (This condition builds up the set T with one element each from each of the sets A_α.)

(ii) If α, β are in Δ and $\alpha \leq \beta$ then $f(\alpha) \leq_T f(\beta)$.

The reader will be able to appreciate the construction if for each real number α, she defines A_α as the interval $[a, \infty)$, Δ as the set of real numbers $\mathbb{R}$ and chooses the usual ordering on $\mathbb{R}$.

Remark. Suppose A, B are sets. In Section 3, we have defined the ordered Cartesian product $A \times B$. Let $\mathfrak{A} = \{A_\alpha, A_\beta\}$, where $A_\alpha = A$, $A_\beta = B$, $\Delta = \{\alpha, \beta\}$ and define $\leq$ on Δ by $\leq = \{(\alpha, \alpha), (\beta, \beta), (\alpha, \beta)\}$. The reader is encouraged to form the product $\prod(\mathfrak{A}, \leq)$ and convince herself that $\prod(\mathfrak{A}, \leq)$ and $A \times B$ are essentially the same (in spite of their different guises). They both capture the same intuitive mathematical idea.

The inevitable need for such an abstract treatment of the ordered products will be transparent if we consider a few examples. For our first example, let us take $A_i = \mathbb{N}$, the set of natural numbers for each i in $\mathbb{N} = \Delta$. Let $\mathfrak{A} = \{A_i / i \in \mathbb{N}\}$. Define $\leq$ as $... \leq 4 \leq 3 \leq 2 \leq 1$. That is, $\leq = \{(i, j) / i, j, i - j \in \mathbb{N}\}$. Then a Δ-tuple in $\prod(\mathfrak{A}, \Delta)$, if written in the form of a list (like a 3-tuple $(2, 5, -3)$ would look like $(..., 5, 4, 3, 2, 1)$. For our second example, let $A_i = \{n_i / n \in \mathbb{N}\}$ for each i in $\Delta = \mathbb{N}$. Define $\leq$ by $1 \leq 3 \leq 5 \leq 7 \leq ... \leq 2 \leq 4 \leq 6 \leq ....$ Let $\mathfrak{A} = \{A_i / i \in \mathbb{N}\}$. Then a typical element of $\prod(\mathfrak{A}, \leq)$ would look like (if written as a list) $(i_1, 3i_2, 5i_3, 7i_4, ..., 2j_1, 4j_2, 6j_3, ...)$ where each i_k and each j_l is a positive integer. As our final example, for each real number i, let A_i denote $\{\sin i, \cos i, e^i\}$. Let $\Delta = \mathbb{R}$, the set of reals. Let the partial order on Δ be the usual ordering of real numbers. Let $=\{A_i / i \in \mathbb{R}\}$. Then $\prod(\mathfrak{A}, \leq)$ can only be understood by means of our definition of $\prod(\mathfrak{A}, \leq)$. That it is not possible to depict Δ-tuples as lists when $\Delta = \mathbb{R}$ has been proved by Georg Cantor[2], a great German mathematician.

[2] See *Resonance*, Vol.19, No.11, 2014.

A Final Example

Consider the product $\mathbb{N} \times \mathbb{N} \times \mathbb{Z} \times \mathbb{N}$. Then what are $\mathfrak{A}, \Delta$ and $\leq$ in this case? Of course, $\mathfrak{A} = \{\mathbb{N}, \mathbb{Z}\}$. For Δ

we may take any set of 4 elements, say $\{a, b, c, d\}$. The partial order $\leq$ is defined as $a \leq b \leq c \leq d$. Also, $A_a = \mathbb{N}, A_b = \mathbb{N}, A_c = \mathbb{Z}, A_d = \mathbb{N}$. In fact, we can imagine a function $A : \Delta \to \mathfrak{A}$ defined by $A(a) = A(b) = A(d) = \mathbb{N}$ and $A(c) = \mathbb{Z}$. Note that $A(a) = A_a, A(b) = A_b, A(c) = A_c$ and $A(d) = A_d$, and the set Δ has more number of elements than $\{A_\alpha/\alpha \in \Delta\}$. Do not entertain the mistaken notion that $\alpha \to A_\alpha$ is a bijection. This is a common misconception among beginners.

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Suggested Reading

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