Sir William Rowan Hamilton

Life, Achievements, Stature in Physics

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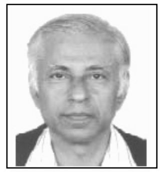
Against the background of the development of physics, and in particular of mechanics, over the centuries since Galileo and Newton, we describe the life and work of William Rowan Hamilton in the 19th century. The depth of his ideas which brought together the understanding of ray optics and classical mechanics, and the remarkable ways in which his work paved the way to the construction of quantum mechanics in the 20th century, are emphasized.



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Introduction

Any student of science, indeed any well-educated person today, is aware that the foundations of modern science rest ultimately on the work and ideas of a small number of larger-than-life figures from the 15th century onwards – Nicolaus Copernicus (1473–1543), Johannes Kepler (1571–1630), Galileo Galilei (1564–1642) and Isaac Newton (1642–1727). Of course the scientific revolution was born out of a much larger social phenomenon – the Renaissance – and many more persons such as René Descartes, Christian Huygens... – were involved. But if one is asked to narrow the choice to as few as possible, these four would be the irreducible minimum. Their great books remain everlasting classics of the era of the birth of modern science – Copernicus' *De revolutionibus orbium coelestium*; Kepler's *Mysterium Cosmographicum* and *Astronomia nova*; Galilei's *Il Saggiatore* and *Dialogue Concerning the Two Chief World Systems*; and finally Newton's *Philosophiae Naturalis Principia Mathematica*, a culmination of this phase of the scientific revolution. Newton's *Principia* was built upon

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and synthesized the earlier ideas of Kepler, Galileo and Descartes on the nature of motion. But of his three Laws of Motion, the third on the equality of action and reaction, and the Law of Universal Gravitation, were uniquely his own.

Among the various branches of modern science it is understandable that it was physics, and within physics the discipline of mechanics, that first achieved a secure mathematical foundation permitting a systematic growth and elaboration in the succeeding centuries. Over the course of the 18th century, the greatest contributors to mechanics were Leonhard Euler (1707–1783), Joseph Louis Lagrange (1736–1813), (known in Italy as Giuseppe Luigi Lagrangia, as they too would like to claim him as their own), and Pierre-Simon Laplace (1749–1827). Lagrange's *Mécanique analytique* and Laplace's *Mécanique céleste* decorate the 18th century as the *Principia* does the 17th.

The pattern of classical mechanics has served as a model for the other major areas in physics, such as electromagnetism, thermodynamics, statistical mechanics, and later the twin theories of relativity followed by quantum mechanics.

Progress in the field of mechanics has been truly impressive over the centuries, with each individual's contributions influenced by and influencing those of many others. Those who have so far been named above are only the most illustrious ones from that period that always spring to mind. The pattern of classical mechanics has served as a model for the other major areas in physics, such as electromagnetism, thermodynamics, statistical mechanics, and later the twin theories of relativity followed by quantum mechanics. In the growth of mechanics itself the interplay between physical ideas and mathematical formulations, with conceptions in the two reinforcing one another, has been of immense importance. From the physics standpoint, the field of optics has grown from the earliest times side-by-side with mechanics, with much give and take.

During the 19th century the greatest contributions to classical mechanics have been from William Rowan Hamilton (1805–1865), Carl Gustav Jacob Jacobi (1804–1851), and somewhat later Henri Poincaré (1854–1912) particularly in the realm of what is called 'qualitative dynamics'. A profoundly critical account of mechanics as viewed from century's end is Ernst Mach's *The Science of Mechanics* published in 1883, which had a deep influence on Albert Einstein.

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Here we celebrate the life and work of Hamilton, in particular the amazing circumstance that so many of his pioneering ideas proved crucial for the creation of quantum mechanics in the 20th century.

A Brief Life Sketch

Hamilton was born at midnight of 3–4 August 1805 in Dublin, Ireland, to Archibald Hamilton, a solicitor by profession, and Sarah Hutton Hamilton. (Their families were originally from England and Scotland.) He was the fourth of nine children. There were many distinguished scientists on his mother's side, suggesting that his scientific genius came from her. While he was still in his teens, Hamilton's mother and father passed away, in 1817 and 1819 respectively.

Hamilton displayed precocious and amazing gifts very early. He could read English by three; knew Greek, Latin and Hebrew by five; and by twelve the major European languages as well as Persian, Arabic, Sanskrit and Hindustani. His interest in mathematics was sparked off at age fifteen. By seventeen he had read both Newton's *Principia* and Laplace's *Mécanique céleste*.

In 1823 he entered Trinity College, Dublin, as an undergraduate, completing his studies there in 1827 and excelling in both science and classics. In 1824 he submitted a paper 'On Caustics' to the Royal Irish Academy. After receiving positive suggestions from (of course) a Committee, he revised and enlarged it by 1827, while still in college, to 'A Theory of Systems of Rays'. This established his reputation. It was here that he presented his concept of characteristic functions in optics. Later there were three supplements, in 1828, 1830 and 1832.

Already in 1827 he was appointed Andrews Professor of Astronomy at Trinity College, and Director of the Dunsink Observatory, five miles away from the centre of Dublin, where he lived for the rest of his life. Designated also as the Astronomer Royal of Ireland, he was spared observational duties, and was left free to concentrate on theoretical work.

Hamilton experienced two early disappointments in romantic relationships – with Catherine Disney in 1824 who married instead a wealthy clergyman fifteen years her senior, but who retained a relationship and correspondence with Hamilton for many years; and with Ellen de Vere around 1830, who however felt she could "not live happily anywhere but at Curragh". Finally he married Helen Maria Bayly in 1833. They had two sons and a daughter, but the marriage was an unhappy one.

The period 1827 to 1834–1835 comprises his 'sunshine years', when he did his finest and most creative work in mathematical physics. He also became close to the poets William Wordsworth and Samuel Taylor Coleridge in England. In 1835 he was knighted, and in 1836 became President of the Royal Irish Academy.

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The range and depth of Hamilton's work are staggering. After the initial work on geometrical optics, based on Fermat's Principle and properties of systems or bundles of rays, he turned to the analogous ideas in dynamics. The highlights here are his version of the Action Principle, and a new and amazingly fruitful form for the equations of motion. The physical content is the same as in Newton's original equations of motion, or in the Euler–Lagrange version of them, but the mathematical shape given to them proved unbelievably powerful. The optico–mechanical analogy is an expression of the similarities in his treatments of geometrical optics and Newtonian particle dynamics. The theory of the Hamilton–Jacobi equations is an especially elegant part of this work. The discovery of quaternions came in 1843; this was more purely mathematical in character, though Hamilton had hopes it could be applied in physics.

The last phase of his life was unfortunately a disaster. Hamilton became an alcoholic, and his home and lifestyle disintegrated completely. Shortly before his death he learned that he had been elected the first Foreign Associate of the newly established United States National Academy of Science. He died on 2nd September 1865 in Dublin, from a severe attack of gout. He had lived all his life in Dublin.

Next, in briefly describing his major achievements and discoveries, the separation into different areas is only for convenience in presentation. There are deep interconnections between them, as must have existed in his own mind. Again for convenience we will use notations and terminology familiar to students today, so that his ideas can be more readily appreciated, and will not strictly follow the chronological order.

The Action Principle

One of the earliest 'minimum' or 'extremum' or 'variational' principles in the era of modern science is Fermat's principle of least time in optics, formulated in 1657. As given by Cornelius Lanczos in his 1949 book *The Variational Principles of Mechanics*,

"The path of a light ray is distinguished by the property that if light travels from one given point M to another given point N, it does so in the smallest possible time."

Thus, in a transparent medium with variable refractive index $n(\mathbf{x})$, and denoting M, N by $\mathbf{x}_1$, $\mathbf{x}_2$ respectively,

“Among all possible motions, Nature chooses that which reaches its goal with the minimum expenditure of action."
– *Maupertuis*

the ray follows a path $\mathbf{x}(s)$ determined by

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} n(\mathbf{x}(s))ds = 0, \quad \mathbf{x}(s_1) = \mathbf{x}_1,\ \mathbf{x}(s_2) = \mathbf{x}_2. \quad (1)$$

We assume an isotropic medium, with refractive index dependent only on position and not on direction.

Here the parameter s measures path length, $(\frac{d\mathbf{x}(s)}{ds})^2 = 1$, and the variation $\delta\mathbf{x}(s)$ of $\mathbf{x}(s)$ leaves the end points fixed.

Probably the earliest analogous idea in mechanics was expressed in 1747 in a somewhat imprecise form by Maupertuis. From Sommerfeld's *Mechanics*,

“Among all possible motions, Nature chooses that which reaches its goal with the minimum expenditure of action.”

Sommerfeld adds:

“This statement of the principle of least action may sound somewhat vague, but is completely in keeping with the form given to it by its discoverer.”

Many others contributed to this stream of thought – d'Alembert with his Principle of Virtual Work, then Euler and Lagrange who succeeded in making Maupertuis' idea more precise. They identified ‘action’ as the time integral of an expression which in the notation familiar today is $p_j \frac{dq^j}{dt}$ – the q^j are coordinates and the p_j are corresponding momenta. However in their formulation it was understood that both the actual and imagined varied motions are energy conserving: $\delta E = 0$. Then in 1834–35, Hamilton gave a new and much more flexible version of the Principle in which the condition $\delta E = 0$ was completely avoided. For conservative systems subject only to holonomic constraints and where forces are derivable from a potential function $V(q)$, Hamilton's Principle says that the actual motion is such that variations about it obey:

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}) dt = 0\,,$$
$$L(q, \dot{q}) = T(q, \dot{q}) - V(q),$$
$$T(q, \dot{q}) = \text{kinetic energy}\,. \tag{2}$$

The dot here denotes the derivative with respect to time.

It is understood that the variations $\delta q^j(t)$ vanish at the terminal times t_1 and t_2, *but are otherwise unrestricted*; and of course $\delta t = 0$ in between, i.e., 'time is not varied'. Thus the actual motion of the system in time is directly characterized, and one immediately obtains from (2) the Euler–Lagrange differential equations of motion (EOM) in time:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} = 0\,. \tag{3}$$

So all the EOM are determined by one single function L. It was in fact Hamilton who gave the name 'Lagrangian' to the integrand in the definition of action above. In this extremely flexible and convenient form, Hamilton's Principle continues to be used in fundamental physics to this day.

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Soon after this work of Hamilton, in the years 1834–37 Jacobi refined and completed this formulation of mechanics, and in 1842–43 gave his celebrated 'Lectures on Dynamics' in Konigsberg, where Hamilton's ideas were carried much further. However, Jacobi dwelt more on mechanics than on optics. (These lectures in English translation have recently been published by Hindustan Book Agency, Delhi.) In particular, Jacobi gave yet another formulation of the Principle of Least Action, in which the dependence of coordinates on time was set aside, and instead, the path of the system in its configuration space was viewed purely geometrically. This version however has not been very fruitful in later times, though in spirit it is close to Fermat's Principle in optics where again time plays no essential role.

In Feynman's well-known three-volume *Lectures on Physics*, Lecture 19 in Volume 2 is a beautiful account of Hamilton's Principle. As Feynman recalls, he had always been fascinated by it, ever since the time in school when his teacher Mr Bader described it to him one day after school hours.

The Canonical Equations of Motion

The Euler–Lagrange EOM (3) can be presented in an interesting form:

$$p_j = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^j}, \quad \dot{p}_j = \frac{\partial L(q, \dot{q})}{\partial q^j}. \tag{4}$$

We regard the first set of equations as definitions of (canonical) momenta, and the second set as the 'true' EOM in the Newtonian sense. From the physical point of view, then, the content of the EOM (4) (or (3)) is the same as of Newton's EOM; however the former are invariant in form under any change in the choice of the (generalized) coordinates q^j. This is also evident from Hamilton's Principle (2). Such changes in q^j are called *point transformations*, and we have the freedom to choose the coordinates to suit the analysis of a given system.

In 1835, Hamilton went one step further and cast the EOM in a remarkably symmetrical form. Treating the q^j and the p_j as independent variables, and defining a function $H(q, p)$ as the Legendre transform of the Lagrangian,

$$H(q, p) = p_j \ \dot{q}^j - L(q, \dot{q}), \tag{5}$$

he obtained the system

$$\dot{q}^j = \frac{\partial H(q, p)}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H(q, p)}{\partial q^j}. \tag{6}$$

These are Hamilton's canonical EOM, they form the basis of his treatment of mechanics. For nonrelativistic systems with kinetic energy quadratic in the velocities,

$H(q, p)$ is the total energy. Equations of this form were apparently used by Lagrange in 1809 in the context of perturbation theory, however it is Hamilton who made them the foundation of dynamics. In his honour, the function $H(q, p)$ is called the Hamiltonian of the system. To avoid misunderstanding, we should mention that we implicitly assume that we are able to express the Lagrangian velocities as functions of the momenta defined in equation (4), and the coordinates. This means that the Lagrangian is nonsingular in the Dirac sense; for singular Lagrangians a comprehensive extension of Hamiltonian dynamics has been created by Dirac.

It is sometimes said that the construction of the phase space, for a given configuration space *Q*, is amongst Hamilton's profoundest discoveries.

The mathematical space on which Hamilton's Principle (2) and the Euler–Lagrange EOM (3) are formulated is the configuration space Q for which q^j are independent coordinates. For the canonical EOM (6) however, the basic space or 'carrier space' is the 'phase space' whose dimension is twice that of Q and for which the q^j and p_j together are independent coordinates. It is sometimes said that the construction of the phase space, for a given configuration space Q, is amongst Hamilton's profoundest discoveries. It turns out that the passage from configuration space Q to its associated phase space is intrinsic and independent of, indeed prior to, the choice of Lagrangian.

Hamilton's treatment of the variables q and p 'on the same footing' leads to a kind of symmetry between them which is profoundly different from the more familiar rotational symmetry among Cartesian spatial coordinates alone coming from Euclidean geometry.

As we said above, the Euler–Lagrange EOM (3) are preserved in form under all point transformations on Q. This then remains true for the canonical EOM (6) as well. However Jacobi then showed that the latter EOM preserve their form – their *Hamiltonian* form – under an *immeasurably larger* group of transformations on

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phase space, the group of all so-called *canonical transformations*. Point transformations form a (very tiny!) subgroup of the group of all canonical transformations. These transformations were not known to Hamilton, and it is to Jacobi that we owe our understanding of their properties and descriptions.

Going beyond this, in a given system of phase space variables q^j, p_j, the solutions in time of EOM of the Hamiltonian form (6) describe the gradual and continuous unfolding of a family of canonical transformations. Thus, the equations expressing $q^j(t_2)$, $p_j(t_2)$ at a later time t_2 in terms of $q^j(t_1)$, $p_j(t_1)$ at an earlier time t_1 constitute a canonical transformation dependent on t_1, t_2 and the system Hamiltonian.

Thus, of the three physically equivalent forms of classical dynamics – the Newtonian, the Euler–Lagrange, and the Hamiltonian forms – it is the last that is the most sophisticated from the structural mathematical point of view. Its profound importance has been further brought out by the circumstance that in both initial forms of quantum mechanics – Heisenberg's matrix mechanics of 1925 and Schrödinger's wave mechanics of 1926 – Hamilton's work is used as the starting point to go from classical to quantum mechanics. Heisenberg's EOM for quantum mechanics are based on (6), converting them into first order differential equations in time for (non-commuting) operators. The Schrödinger wave equation starts from the classical Hamiltonian $H(q, p)$, constructs a linear operator out of it capable of acting on complex wave functions $\psi(q)$ by the rule of replacement $p \to -i\hbar\frac{\partial}{\partial q}$, and then sets up the time dependent Schrödinger wave equation. Many years after Feynman learned about Hamilton's Principle of Least Action from Mr Bader, he discovered a third form of quantum mechanics called the Path Integral form. This was based on work by Dirac in 1934 on the role of the Lagrangian in quantum mechanics, and in it Feynman made essential use of the action

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as defined by Hamilton in equation (2). It is awe inspiring to realise that Hamilton's work around 1834–35 played such a crucial role in the birth of quantum mechanics nine decades later! Yet another aspect of this will be described below.

Optico-mechanical analogy and the Hamilton–Jacobi Equations

The presence of analogies between the descriptions of optical and mechanical phenomena was sensed quite early, by John Bernoulli and Maupertuis. As we have seen, Hamilton's work on geometrical optics preceded his work on mechanics. There were four memoirs on optics, in the years 1827, 1828, 1830 and 1832, in which he developed the idea of *characteristic functions* as a basis for geometrical optics, based on Fermat's Principle (1). (The quite remarkable prediction of conical refraction came in the fourth memoir.) Even though it may be a bit demanding, we sketch these ideas here.

The article by R Nityananda in this issue of *Resonance* builds up to Hamilton's optics from the principles of Fermat and Huygens.

Fermat's Principle (1) leads to a system of second order ordinary differential equations for rays in geometrical optics:

$$\begin{aligned} &n(\mathbf{x})\ddot{x}_j = (\delta_{jk} - \dot{x}_j\dot{x}_k)\partial_k n(\mathbf{x})\,, \\ &\dot{x}_j\dot{x}_j = 1, \quad \dot{x}_j\ddot{x}_j = 0\,, \quad j,k = 1,2,3. \end{aligned} \tag{7}$$

Now the dots signify derivatives with respect to distance s along the ray, not with respect to time.

For any choice of 'initial data' $x_j(s_1), \dot{x}_j(s_1)$ we get one definite ray $\mathbf{x}(s)$. Alternatively, we may choose $\mathbf{x}_1 = \mathbf{x}(s_1)$ and $\mathbf{x}_2$ independently; then in the generic case we get a ray $\mathbf{x}(s)$ and a value for s_2 such that $\mathbf{x}(s_2) = \mathbf{x}_2$. (We cannot choose s_2 independently since $\dot{\mathbf{x}}(s)^2 = 1$ must be obeyed!)

The *point characteristic* of Hamilton is then defined as

$$V(\mathbf{x}_1, \mathbf{x}_2) = \int\limits_{\substack{s_1 \\ \text{along ray}}}^{s_2} ds\, n(\mathbf{x}(s)),\ \mathbf{x}(s_1) = \mathbf{x}_1, \mathbf{x}(s_2) = \mathbf{x}_2\,. \tag{8}$$

This determines the 'directional' properties of the ray from $\mathbf{x}_1$ to $\mathbf{x}_2$:

$$\boldsymbol{\nabla}_{\mathbf{x}_2} V(\mathbf{x}_1, \mathbf{x}_2) = n(\mathbf{x}_2)\dot{\mathbf{x}}_2, \boldsymbol{\nabla}_{\mathbf{x}_1} V(\mathbf{x}_1, \mathbf{x}_2) = -n(\mathbf{x}_1)\dot{\mathbf{x}}_1. \tag{9}$$

These are the basic equations of Hamiltonian optics, at the level of single rays.

Other kinds of characteristic functions in optics were defined by Hamilton as various Legendre transforms of the point characteristic $V(\mathbf{x}_1, \mathbf{x}_2)$.

Next we come to systems or families of rays, families of solutions to (7), built up in a very special manner. While (7) is a system of ordinary differential equations for $\mathbf{x}(s)$, now we have to deal with a partial differential equation for a real function $S(\mathbf{x})$ on space:

$$|\boldsymbol{\nabla} S(\mathbf{x})| = n(\mathbf{x})\,. \tag{10}$$

(Many years later this equation acquired the name 'eikonal equation', and $S(\mathbf{x})$ the 'eikonal'.) For a given $S(\mathbf{x})$, the choice of initial data

$$\mathbf{x}(s_1) = \mathbf{x}_1, \quad \dot{\mathbf{x}}(s_1) = \frac{1}{n(\mathbf{x}_1)}(\boldsymbol{\nabla} S(\mathbf{x}))_{\mathbf{x}=\mathbf{x}_1}\,, \tag{11}$$

leads via (7) to a definite ray $\mathbf{x}(s)$. Analysis shows that, since $S(\mathbf{x})$ obeys (10), $\mathbf{x}(s)$ is determined by a system of first order ordinary differential equations:

$$n(\mathbf{x}(s))\dot{\mathbf{x}}(s) = (\boldsymbol{\nabla} S(\mathbf{x}))_{\mathbf{x}=\mathbf{x}(s)}\,, \ \mathbf{x}(s_1) = \mathbf{x}_1\,. \tag{12}$$

Allowing the initial point $\mathbf{x}_1$ to vary over a suitably chosen two-dimensional 'transverse region' in physical space, for example a surface $S(\mathbf{x}) =$ constant, we get a family or bundle of rays determined by $S(\mathbf{x})$, filling out some three-dimensional region in space.

In this picture, *wave fronts* are defined as surfaces of constant $S(\mathbf{x})$, while *rays* (within the family!) are trajectories orthogonal to the wave fronts. Again within the family corresponding to $S(\mathbf{x})$, Hamilton's point characteristic (8) becomes

$$V(\mathbf{x}_1, \mathbf{x}_2) = S(\mathbf{x}_2) - S(\mathbf{x}_1)\,. \qquad (13)$$

Hamilton then transferred these ideas to mechanics. Now time enters as an independent variable, leading to some important changes. The discussion becomes 'dynamical', and not simply geometrical as in optics.

The canonical EOM (6) can be solved for given definite data at an initial time t_1, leading to a definite phase space trajectory:

$$q^j(t_1), p_j(t_1)\,\text{at}\,t_1 \rightarrow q^j(t), p_j(t)\ \ \text{for}\,t \geq t_1\,. \qquad (14)$$

Alternatively we may specify a solution by giving some initial and some final data:

$$t_1, q^j(t_1), t_2, q^j(t_2) \rightarrow q^j(t), p_j(t)\ \ \text{for}\,t_1 \leq t \leq t_2\,. \qquad (15)$$

It now turns out that the action appearing in Hamilton's Principle (2), *evaluated for the solution* (15) *of the canonical EOM*, is the analogue of the point characteristic $V(\mathbf{x}_1, \mathbf{x}_2)$ in (8) in optics:

$$S(q(t_1), t_1; q(t_2), t_2) = \underset{\text{along trajectory}}{\int_{t_1}^{t_2}} dt\, L(q(t), \dot{q}(t)); \qquad (16a)$$

$$\begin{aligned} p_j(t_2) &= \frac{\partial S(q(t_1), t_1; q(t_2), t_2)}{\partial q^j(t_2)}, \\ p_j(t_1) &= -\frac{\partial S(q(t_1), t_1; q(t_2), t_2)}{\partial q^j(t_1)}\,. \end{aligned} \qquad (16b)$$

This is called *Hamilton's Principal function*. It acts in the sense of (16b), as the *Generating Function* for the canonical transformation connecting $q^j(t_2), p_j(t_2)$ to $q^j(t_1), p_j(t_1)$. That every canonical transformation can be described in this way (or a variant thereof) via a Generating Function was shown by Jacobi. We should also mention a beautiful extension of Jacobi's result by Constantin Carathéodory many years later.

This function obeys two partial differential equations with respect to the time variables:

$$\begin{aligned}\frac{\partial S}{\partial t_2} + H\left(q(t_2), \frac{\partial S}{\partial q(t_2)}\right) &= 0, \\ \frac{\partial S}{\partial t_1} - H\left(q(t_1), \frac{\partial S}{\partial q(t_1)}\right) &= 0, \\ S = S(q(t_1), t_1; q(t_2), t_2) &\text{ throughout.}\end{aligned} \qquad (17)$$

These are the famous Hamilton–Jacobi (H–J) partial differential equations of mechanics.

The general H–J problem is the search for a solution $S(q, t)$ to the partial differential equation

$$\frac{\partial S(q,t)}{\partial t} + H\left(q, \frac{\partial S(q,t)}{\partial q}\right) = 0\,, \text{given } S(q, t_1) = S_1(q)\,, \qquad (18)$$

presented as an initial value problem. This is the mechanics analogue to (10) in optics. The solution to (18) is called a Hamilton Principal function. The link to phase space trajectories obeying the canonical EOM (6) is as follows: Since $S(q, t)$ obeys (18), the canonical EOM simplify to just the set

$$\frac{dq^j}{dt} = \left(\frac{\partial H(q,p)}{\partial p_j}\right)_{p=\frac{\partial S(q,t)}{\partial q}}\,,\ q = q(t)\,. \qquad (19)$$

Thus $S(q, t)$ determines a special family of phase space trajectories, corresponding to the selected set of initial conditions $\left(q^j, p_j = \frac{\partial S_1(q)}{\partial q^j}\right)$ at time t_1, with q varying over configuration space Q.

Let us mention that since we have (implicitly) assumed that $H(q,p)$ has no explicit time dependence, there is a time independent version of the H–J equation, whose solutions are called Hamilton characteristic functions.

The Hamilton–Jacobi equations appear also in the semi-classical approximation to the Schrödinger wave equation of quantum mechanics.

It is widely felt that the Hamilton–Jacobi equation is the most beautiful form of classical mechanics. (Remember though that it is a single partial differential equation, unlike the Euler–Lagrange or Hamiltonian EOM (3,4,6).) At the start of his discussion of this equation, Lanczos places this quotation from the old testament:

It is widely felt that the Hamilton–Jacobi equation is the most beautiful form of classical mechanics.

"Put off thy shoes from off thy feet, for the place whereon thou standest is holy ground." (Exodus III, 5)

Hamilton's optico-mechanical analogy was a great source of inspiration to Schrödinger in the creation of wave mechanics. As for the construction of special families of rays in optics or of phase space trajectories in mechanics we must quote an eloquent passage from Paul Dirac. In his paper titled 'The Hamiltonian form of field dynamics' (*Canadian Journal of Mathematics*, Vol.3, p.1, 1951) he says:

"In classical dynamics one has usually supposed that when one has solved the equations of motion one has done everything worth doing. However, with the further insight into general dynamical theory which has been provided by the discovery of quantum mechanics, one is led to believe that this is not the case. It seems that there is some further work to be done, namely to group the solutions into families (each family corresponding to one principal function satisfying the Hamilton–Jacobi

The algebra of quaternions and some applications to geometry and group theory are discussed in the article by G S Krishnaswami and S Sachdev in this issue of *Resonance*.

equation). The family does not have any importance from the point of view of Newtonian mechanics; but it is a family which corresponds to one state of motion in the quantum mechanics, so presumably the family has some deep significance in nature, not yet properly understood."

Quaternions

Hamilton's creation of the algebra of quaternions came at a later stage in his life, after the work in optics and mechanics. Complex numbers were known, and could be represented on a Euclidean plane. Hamilton's idea was to extend this to higher spatial dimensions, so as to be applicable in physics. After much effort he finally succeeded in 1843, though now it is known that Benjamin Olinde Rodrigues had found essentially the same results in 1840. In place of the single pure imaginary unit i for complex numbers, quaternions are built using three such units i, j and k. Their essential algebraic or composition properties became suddenly clear to Hamilton during a walk with his wife along the Royal Canal from Dunsink to a meeting of the Royal Irish Academy on 16th October 1843:

$$i^2 = j^2 = k^2 = ijk = -1\,. \tag{20}$$

"I then and there felt the galvanic circuit of thought close, and the sparks which fell from it were the fundamental equations between *i, j, k*; exactly such as I have used them ever since."
– *Hamilton*

He is said to have immediately carved these formulae with his penknife on the stone of Broome Bridge as he passed it. In a letter to Peter Guthrie Tait many years later, on 15th October 1858, he described what had happened in these words:

"I then and there felt the galvanic circuit of thought close, and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them ever since."

Indeed he went on to claim: "I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions (the calculus) was for the close of the seventeenth".

In 1958 on behalf of the Royal Irish Academy the Irish mathematician-political leader Eamon de Valera unveiled a plaque under Broome Bridge to commemorate the event.

Hamilton wrote extensively on quaternions, his *Lectures on Quaternions* being published in 1853. They are the first example within mathematics of algebraic quantities obeying a non-commutative but associative rule of multiplication. Recall here that physical quantities are represented in quantum mechanics by generally non-commuting operators. One little gem in this book is Hamilton's construction of a pictorial representation for elements of the group SU (2) and their (non-commutative) composition law, using great circle arcs on a sphere S^2. These 'turns' of Hamilton have been used in physics in recent times.

However, quaternions did not prove as useful for physics as Hamilton had hoped. The methods of vector algebra and vector calculus later pioneered by Oliver Heaviside and Josiah Willard Gibbs turned out more useful and capable of extension to any number of dimensions.

Concluding Remarks

This account has hopefully succeeded in conveying to the reader the depth and profound beauty of Hamilton's contributions to theoretical physics. His exalted status among the greatest physicists of all time rests on his magnificent achievements. There is no better way to conclude than by quoting from two of his countrymen, one belonging to his times and the other from a few generations later:

"Hamilton was gifted with a rare combination of those qualities which are essential instruments of discovery. He had a fine perception by which the investigator is guided in his passage from the known to the unknown....

But he seems, also, to have possessed a higher power of divination – an intuitive perception that new truths lay in a particular direction, and that patient and systematic search, carried on within definite limits, must certainly be rewarded by the discovery of a path leading into regions hitherto unexplored...."

– Memorial address, November 1865, by Charles Graves, President Royal Irish Academy.

'Hamilton lived in the heroic age of mathematics.... It was not Hamilton's ambition to polish the corners of structures built by other men. Newton had the theory of gravitation and planetary motions, Lagrange had his dynamical equations, Laplace had the theory of potential. What monument would Hamilton create to make his memory imperishable? Hamilton realised that optics and dynamics are essentially a single mathematical subject. He was able to characterise or describe any optical or dynamical system by means of a single characteristic or principal function.... Hamilton liked to refer to himself in the words Ptolemy used of Hipparchus: a lover of labour and a lover of truth...."

–John Lighton Synge, 1943.

Suggested Reading

[1] **Ernst Mach, *The Science of Mechanics*: *A Critical and Historical Account of its Development*, Open Court Publishing Company, La Salle, Illinois, 6th Edition, 1988, (originally published 1883).**

[2] **Cornelius Lanczos, *The Variational Principles of Mechanics*, Dover Publications, New York, 4th Edition, 1986, (originally published 1949).**

[3] **Arnold Sommerfeld, *Lectures on Theoretical Physics*, Vol.1: *Mechanics*, Academic Press, New York 1952; Levant Books, Kolkata, 2006.**

[4] ***Jacobi's Lectures on Dynamics* – 2nd Edition, Edited by A Clebsch, Translated by K Balagangadharan, Hindustan Book Agency, Delhi, 2009.**

[5] **R P Feynman, R B Leighton and M Sands, *The Feynman Lectures on Physics*, Addison Wesley 1964, Vol.2, Lecture 19.**

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