On the Prime Ideals of $C[0,1]$

Vaibhav Pandey

It is well known that the set $C[0,1]$ of real-valued continuous functions on the closed interval $[0,1]$ has a natural ring structure. Its maximal ideals are known to be points. Here we show that although there exist prime ideals which are not maximal, the zero set is again a singleton. In particular, each prime ideal is contained in a unique maximal ideal.

Vaibhav Pandey is a final year Integrated MSc student in mathematics at NISER, Bhubaneswar. He is interested in pursuing research in algebra and exploring the links between algebra and geometry.

Introduction

A basic object of study in real analysis is the set $C[0,1]$ of all continuous real-valued functions defined on the closed interval $[0,1]$. It is often profitable to equip such a set with additional structure and study its properties. One such aspect is to view the set algebraically. More precisely, the set $C[0,1]$ has the structure of a commutative ring under point-wise addition and multiplication; that is,

$$(f+g) : x \mapsto f(x) + g(x) \ ,$$
$$(f.g) : x \mapsto f(x)g(x).$$

The fact that functions and polynomials can be added and multiplied in this manner allows us to use the theory of rings to obtain analytic information. In commutative algebra or in algebraic geometry, two of the most important notions are those of prime ideals and of maximal ideals in a ring. For instance, the article ([1]) of V Pati in this journal has a comprehensive introduction to the fundamental theorem known as Hilbert's Nullstellensatz (the theorem of zeros). One version of the Nullstellensatz is the assertion that for a set S of polynomials in n variables over the complex numbers, there exists an n-tuple of complex numbers $(a_1, \cdots, a_n)$ where all the

Keywords
Ring of continuous functions, prime ideals, maximal ideals, zero sets.

The nullstellensatz of Hilbert can be viewed as the higher-dimensional generalization of the fundamental theorem of algebra.

polynomials have value zero, unless the following exception occurs. The exception is when $1 = g_1 f_1 + \cdots + g_r f_r$ where the f_i's come from S and g_i's are some polynomials. In the one variable case $n = 1$, this is the so-called fundamental theorem of algebra. The main part of the proof of the Nullstellensatz is to show that every maximal ideal of the ring $\mathbf{C}[X_1, \cdots, X_n]$ of complex polynomials in n variables is of the form

$$(X_1 - a_1, \cdots, X_n - a_n)$$

for some $(a_1, \cdots, a_n) \in \mathbf{C}^n$. Here, we use the notation $(f_1, \cdots, f_r)$ for the ideal generated by the f_i's; that is, it consists of all the polynomials of the form $g_1 f_1 + \cdots + g_r f_r$ as g_i's vary over the set of all polynomials. The theorem of zeros is a deep, though basic, result of algebraic geometry. The ring $C[0, 1]$ is not as nice as the above polynomial rings, but is still amenable to a fruitful algebraic treatment. For basic notions like ring homomorphisms and prime ideals, refer to [1] or to any standard text in algebra like [2].

Ring-Theoretic Properties of $C[0, 1]$

We start by recalling that a proper ideal M of a commutative ring A with unity, is maximal if and only if the quotient ring A/M is a field. Similarly, a proper ideal P is prime if and only if the quotient ring A/P is an integral domain (that is, has no zero divisors). So, maximal ideals are always prime but the converse is not true (for example, (0) is a prime ideal that is not maximal in $\mathbf{Z}$). We first list some ring theoretic properties of $C[0, 1]$ (see also [2]):

We may consider any two nonzero functions *f* and *g* in *C*[0,1] whose zero sets are complements of each other in [0,1]; then the product *f g* is the zero function.

• $C[0, 1]$ *is not an integral domain; that is, there exist functions* $C[0, 1]$ *which are different from the zero function and whose product is the zero function.*

In fact, we may consider any two nonzero functions f and g in $C[0, 1]$ whose zero sets are complements of each other in $[0, 1]$; then the product fg is the zero function.

So, $C[0,1]$ is not an integral domain.

• *Every maximal ideal of* $C[0,1]$ *is a point*; *more precisely, for* $a \in [0,1]$, *the sets*

$$M_a := \{f \in C[0,1] : f(a) = 0\}$$

are precisely the maximal ideals of $C[0,1]$.

The fact that maximal ideals of C[0,1] are points is a consequence of the compactness property of [0,1].

This statement which is the analogue of Hilbert's Nullstellensatz is much easier to prove though. Note that for any a, the set M_a is a maximal ideal – it is clearly an ideal and the ring homomorphism

$$\theta_a : C[0,1] \to \mathbf{R} \; ; \; \theta_a(f) = f(a)$$

takes all real values and has kernel equal to M_a. The converse statement made above requires a property of the closed interval $[0,1]$ which is known in general by the name of compactness. This implies the property that if open subintervals U_a of $[0,1]$ are chosen containing each point $a \in [0,1]$ (where open intervals around 0 and 1 inside $[0,1]$ are understood as $U_0 = [0,\epsilon)$ and $U_1 = (1-\delta, 1]$ for some $\epsilon, \delta > 0$), then there are finitely many points $a_1, \cdots, a_n$ in $[0,1]$ so that

$$[0,1] = U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}.$$

Now, let M be any maximal ideal of $C[0,1]$.

Suppose, if possible, that no point of $[0,1]$ is a common zero for all the functions in M. Then, if $a \in [0,1]$, there exists $f_a \in M$ such that $f_a(a) \neq 0$. As f_a is continuous, it follows that f_a is not zero in an open interval around a; that is,

$$\{b \in [0,1] : f_a(b) \neq 0\}$$

contains an open interval U_a containing a. Note that this is valid for $a = 0$ and $a = 1$ also, where the meaning of open intervals around 0 and 1 is as above. So,

$$[0,1] = \bigcup_{a \in [0,1]} U_a \, ,$$

which gives finitely many $a_1, \cdots, a_n \in [0,1]$ such that

$$[0,1] = \cup_{i=1}^n U_{a_i}.$$

Consider the function $f = \sum_{i=1}^n f_{a_i}^2 \in M$. It cannot have a zero in $[0,1]$ for, if a were a zero for this function, then $f_{a_i}(a) = 0$ for all $i = 1, \cdots, n$ so that $a \notin \bigcup_{i=1}^n U_{a_i} = [0,1]$ which is impossible. Thus, $\frac{1}{\sum_{i=1}^n f_{a_i}^2}$ are continuous functions and, therefore, so are $g_j := \frac{f_{a_j}}{\sum_{i=1}^n f_{a_i}^2}$ for $j \leq n$. Noting that

$$g_1 f_{a_1} + \cdots + g_n f_{a_n} = 1\,,$$

it follows that $1 \in M$ and $M = C[0,1]$ which contradicts the fact that it is a proper ideal. Thus, we must have a common zero $a \in [0,1]$ for all elements of M. Then $M \subset M_a$. As both are maximal, we must have $M = M_a$.

As a consequence, we have:

• *The intersection of all maximal ideals of* $C[0,1]$ *is zero.*

Indeed, this intersection consists of functions which vanish at every point, and must therefore be only the zero function.

• $C[0,1]$ *has arbitrarily long, strictly increasing chains of proper ideals* (*i.e., it is not a Noetherian ring*).

In fact,

$$(x) \subset (x^{1/3}) \subset (x^{1/3^2}) \subset (x^{1/3^3}) \cdots\cdots$$

is a strictly increasing chain of ideals. More generally,

$$(0) = I_2 \subset I_3 \subset I_4 \subset I_5 \subset \cdots\cdots$$

The ring C[0,1] has arbitrarily long increasing sequences of ideals.

is also a strictly increasing chain of ideals where

$$I_k = \{f \in C[0,1] : f(x) = 0 \,\forall\, x \in [1/2 - 1/k, 1/2 + 1/k]\}.$$

• *There exist prime ideals in $C[0,1]$ which are not maximal*

For the proof, we need the notion of a multiplicative subset in a commutative ring with unity.

DEFINITION

A nonempty set S in a commutative ring A with unity is said to be multiplicative if: (i) $1 \in S, 0 \notin S$ and (ii) given $s, t \in S$, the product $st \in S$.

Let us now prove the existence of non-maximal prime ideals in the ring $C[0,1]$.

Let S be the set of all nonzero polynomial functions in $C[0,1]$. Note that S is a multiplicative set. Now consider all the ideals in $C[0,1]$ with the property that they are disjoint from the subset S. Call this set A with the partial ordering of set inclusion. Note that A is nonempty since the zero ideal belongs to it. Consider a chain $I_1 \subset I_2 \subset \ldots I_n \ldots$ of ideals in A. Then the ideal $\bigcup I_j$ is clearly an upper bound. By Zorn's Lemma, A has a maximal element (note that there may be many maximal elements). Let P be any maximal element.

We claim that P is a prime ideal. If not, then there exist $a, b \in C[0,1]$ (outside P) such that $ab \in P$. Consider the ideals (P, a) and (P, b). Both these ideals strictly contain P and therefore must intersect with S. Hence, there exist $f, g \in R$ and $p, p' \in P$ such that $p + fa, p' + gb \in S$. As S is multiplicative, $(p + fa)(p' + gb) \in S$. Now $(p + fa)(p' + gb) = pp' + pgb + p'fa + fgab$. As P is an ideal, therefore $pp', pgb, p'fa \in P$. By assumption $ab \in P$, so $fgab \in P$. Therefore, we get that $(p + fa)(p' + gb) \in P$. But this is a contradiction to the fact that $S \cap P = \emptyset$. Hence, P is a prime ideal and the claim is proved.

We prove by contradiction that P is not a maximal ideal. If P were a maximal ideal, then we would have $P = M_\gamma$ for some $\gamma \in [0,1]$. Then the nonzero polynomial $p(x) =$

$x - \gamma \in P$. But p is in S also; this contradiction proves that P is not a maximal ideal.

A Question on the Zero Set

We know that all maximal ideals of $C[0,1]$ are of the form M_γ for some $\gamma \in [0,1]$, where

$$M_\gamma = \{f \in R : f(\gamma) = 0\}.$$

In other words, if for any subset $S \subset C[0,1]$, we define

$$V(S) := \{a \in [0,1] : f(a) = 0 \ \forall \ f \in S\},$$

then $V(M_\gamma) = \{\gamma\}$. We ask:

QUESTION

For a prime ideal P of $C[0,1]$, how large can $V(P)$ be?

An Observation

We observe that given $\gamma_1 \neq \gamma_2 \in [0,1]$, the ideal $I = \{f \in R : f(\gamma_1) = f(\gamma_2) = 0\}$ is not prime. This is because the polynomial $(x-\gamma_1)(x-\gamma_2) \in I$, but neither $x - \gamma_1$ nor $x - \gamma_2$ belongs to I. From the above discussion, it appears that ideals vanishing at 2 or more points are not prime ideals. Note that for each ideal I, the subset $V(I) \subset [0,1]$ is closed and bounded (and hence, compact) in $[0,1]$. Also note that if $I_1 \subset I_2$, then $V(I_2) \subset V(I_1)$. We answer the above question as follows.

Theorem. *If P is a prime ideal of $C[0,1]$, then $V(P)$ is a singleton.*

Every prime ideal of $C[0,1]$ is contained in a unique maximal ideal.

Proof. Since $C[0,1]$ is a ring with 1, any proper ideal I is contained in a maximal ideal M_γ for some $\gamma \in [0,1]$. Then, $\gamma \in V(I)$; so, $|V(I)| = 0$ is not possible for any proper ideal I. Now consider the case when $|V(P)| \geq 2$. Let $\gamma_1, \gamma_2 \in V(P)$ with $\gamma_1 < \gamma_2$. Find points x_1 and x_2

in (γ_1, γ_2) such that $x_1 < x_2$. Define

$$f(x) = \begin{cases} 0 & x \leq x_2 \\ x - x_2 & x > x_2 \end{cases}$$

and

$$g(x) = \begin{cases} x - x_1 & x \leq x_1 \\ 0 & x > x_1 \end{cases}$$

Then, $f(\gamma_2) \neq 0$ and $g(\gamma_1) \neq 0$; so f and g do not belong to I whereas $fg = 0$ belongs to I. Thus, I is not a prime ideal. Note that the above argument works equally well regardless of whether $V(P)$ is a finite set or an infinite set. In other words, $|V(P)| = 1$ for every prime ideal P. □

We now note an interesting consequence:

Corollary

Any prime ideal of $C[0, 1]$ *is contained in a unique maximal ideal.*

We showed an existential proof for prime ideals that are non-maximal and showed that the zero set does not distinguish them. It would be a good follow-up if a constructive proof or a set of generators can be given for a non-maximal prime ideal in the ring $C[0, 1]$. It is known that the maximal ideals of $C[0, 1]$ are actually uncountably generated ([3], p. 404) and it is probably difficult to come up with a generating set for non-maximal prime ideals in this ring.

Acknowledgments

I am deeply thankful to Prof. B.Sury (ISI, Bangalore) with whom I discussed this problem and got useful insights. His help in giving shape to this article was also greatly benefiting. I would also like to thank my teachers- Dr. B.K.Sahoo, Dr. Amit Tripathi and Dr. Tanusree Khandai (all from NISER, Bhubaneswar). Long discussions with them helped me refine my arguments and get a better perspective of the subject. I am always thankful to my parents for their unconditional support and appreciation.

Address for Correspondence
Vaibhav Pandey
National Institute of Science Education and Research Bhubaneswar
P.O. Jatni, Khurda 752050, Odisha, India
Email: vaibhav2011@niser.ac.in

Suggested Reading

[1] V Pati, Hilbert's Nullstellensatz and the beginning of Algebraic Geometry, *Resonance*, Vol.4, No.8, pp.36–57, 1999.

[2] I N Herstein, *Topics in Algebra*, 2nd Ed., John Wiley & Sons, Inc., New York, 1975.

[3] B Sury, Uncountably generated ideals in C[0,1], *The College Mathematics Journal*, Vol.42, No.5, pp.404–406, November, 2011.