In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Deriving a Unified Equation for Doppler Effect for any Wave in any Medium from Lorentz Transformations

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We use the Lorentz transformation equations to derive a unified equation for the Doppler effect – that can be used for any one-dimensional (sound, water, EM, etc.) wave in any medium. This unified equation includes the effects of motion, if present, of the medium relative to the observer as well as the relative velocity between the observer and the source. This master equation can be applied to both relativistic and non-relativistic situations to recover the more familiar Doppler effect expressions and it clarifies that the Doppler equations given in standard textbooks for both sound and EM waves, are basically the same. The advantage of this unified equation is that it reduces the effort in solving complex problems. Basic knowledge of Lorentz transformations and their physical effects is enough to understand this derivation.

Keywords
Doppler effect, Lorentz transformations, EM waves, sound waves.

1. Introduction

Sometimes, students may feel confused when they encounter two entirely different equations for describing

We introduce a master equation which lessens the effort to solve problems.

the Doppler effect in relativistic and non-relativistic cases. Many standard textbooks introduce the Doppler effect by stating these two different equations – one for sound and other for light in vacuum. They look completely different and students may fail to realize that these equations are basically two different forms of a master equation which we will derive in a later section. Also, in the case of sound, the equation we normally encounter in textbooks is not too easy to use to solve problems, especially when the motion of the medium is also to be considered. So, we introduce a master equation which lessens the effort to solve problems. It is possible to apply this equation in relativistic and non-relativistic cases to recover the familiar equations.

1.1 *Non-relativistic Waves*

In the case of non-relativistic waves, such as water waves, sound waves, etc., we recover a more general form of the familiar Doppler formula (1) which includes the effect of the motion of the medium relative to the observer.

$$f' = f\frac{v_{wm} + v_{mo}}{v_{wm} - v_{sm}}. \tag{1}$$

Here v_{ab} is the velocity of a relative to b, and $w \equiv$ wave, $m \equiv$ medium, $o \equiv$ observer and $s \equiv$ source. Also, $v_{ab} = -v_{ba}$. Since we are restricting ourselves to one-dimensional motion, v_{ab} is taken to be positive if it is in the $+x$ direction and negative if it is in the $-x$ direction.

1.2 *Relativistic Waves*

In the case of relativistic waves such as EM waves in vacuum and medium, we not only recover the familiar Doppler formula (2) for light in vacuum from our master equation, but also can account for the change in the formula when the EM waves have to propagate in a medium with a velocity (relative to observer) different from c.

$$f' = f\sqrt{\frac{1 + v/c}{1 - v/c}}, \tag{2}$$

where v is the relative velocity of the source with respect to the observer.

Part of the reason for the origin of two different expressions (1) and (2) has to do with the fact that the velocities in (1) are defined with respect to the medium, whereas in (2), they are defined relative to the observer. When we define the velocities relative to the medium, it is difficult to solve problems especially when the medium itself is in motion relative to the observer. In such cases, it is convenient to use the master equation, which we present, to solve the problem without much effort.

Lorentz transformations are the coordinate transformation equations which accommodate relativistic effects such as length contraction and time dilation.

The derivation of the master equation from Lorentz transformation is straightforward. We derive the required equation considering the relative velocities of the source and the wave with respect to the observer. Any motion of the medium is absorbed in the relative velocity of the wave using Einstein's velocity addition formula[1]. We do not make any assumptions regarding the nature of the wave and the medium during the derivation. This derivation helps us to understand the conditions for the validity of (1) and (2), and allows us to appreciate the reason for the existence of two different expressions for the Doppler effect. Since we derive this expression from Lorentz transformations, we accommodate all the relativistic effects so that it will not lack any generality and can be used in any situation.

[1] If we have two frames of reference S and K where K is moving with respect to S with a velocity of V_{SK}, then a moving particle of velocity V_{SP} (relative to S) will have a different velocity V_{KP} relative to the K frame. They are connected by Einstein's velocity addition formula as:

$$\frac{V_{SK}+V_{KP}}{1+\frac{V_{SK}.V_{KP}}{c^2}}=V_{SP}$$

where c is the speed of light in vacuum.

We first recall the basics of Lorentz transformations and then we describe how to derive the required master equation using the Lorentz transformation equations. Then, we apply this master equation to various special cases to recover the familiar Doppler effect expressions.

2. Lorentz Transformations

Lorentz transformations are the coordinate transformation equations which accommodate relativistic effects such as length contraction and time dilation when we

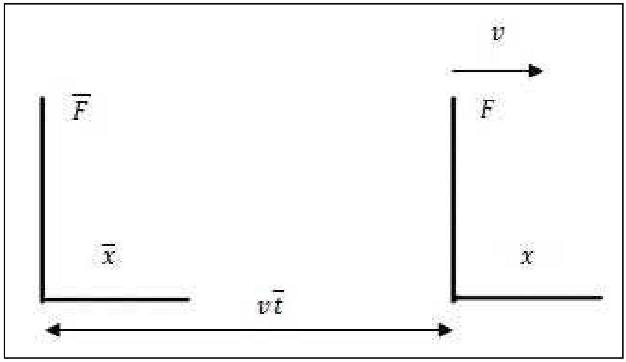

Figure 1. Frame of reference F is moving relative to $\bar{F}$ with velocity v.

transform the coordinates from one frame of reference to another. Consider two frames of reference F and $\bar{F}$ (see *Figure* 1).

Here, F is moving with velocity v relative to $\bar{F}$. According to Lorentz transformations, we have the following equations:

$$x = \gamma\left(\bar{x} - v\bar{t}\right), \tag{3}$$

$$t = \gamma\left(\bar{t} - \frac{v\bar{x}}{c^2}\right), \tag{4}$$

$$\bar{x} = \gamma(x + vt), \tag{5}$$

$$\bar{t} = \gamma\left(t + \frac{vx}{c^2}\right). \tag{6}$$

Here, (3) and (4) transform coordinates from $\bar{F}$ to F, and (5) and (6) do the opposite transformation. In the following section, we utilize these transformation equations to derive our master equation for the Doppler effect.

3. Deriving a Master Equation for the Doppler Effect

When $t = \bar{t} = 0$, the origins of the two reference frames coincide, i.e., $x = \bar{x} = 0$. After time $\bar{t}_1$ (it is to be understood that physical quantities with an upper bar are measured in the $\bar{F}$ frame), the F frame is at a distance of $v\bar{t}_1$ from $\bar{F}$. Suppose the first crest of the wave is emitted from $\bar{x} = 0$ at this instant of time. That is, this event has the coordinates $\bar{x} = \bar{x}_1 = 0$ and $\bar{t} = \bar{t}_1$. Using (3) and (4), the coordinates of this event with respect to F are

$$\begin{aligned} x_1 &= -\gamma v \bar{t}_1, \qquad (7)\\ t_1 &= \gamma \bar{t}_1 \ . \qquad (8) \end{aligned}$$

The first crest has to move a distance of

$$|x_1| = |\gamma v \bar{t}_1| \qquad (9)$$

to reach the observer in the F frame having the coordinates $x = 0$ and $t = t_1$. The time taken by this crest to reach $x = 0$ is

$$\mu_1 = \frac{|x_1|}{|v_{wo}|}, \qquad (10)$$

where $|v_{wo}|$ is the effective wave speed relative to the observer, since the wave velocity is independent of the source velocity. The first crest will reach the observer at time

$$t_1' = t_1 + \mu_1 \ . \qquad (11)$$

The wave velocity has to be positive for the wave to reach the observer who is in the $+x$ direction relative to the source, and henceforth we will drop the distinction between wave velocity and wave speed. Next, after a time $\bar{T}$, consider the emission of the second crest from $\bar{F}$ with event coordinates $\bar{x} = \bar{x}_2 = 0$ and $\bar{t} = \bar{t}_2 = \bar{t}_1 + \bar{T}$. The coordinates of this event with respect to F are

$$\begin{aligned} x_2 &= -\gamma v \bar{t}_2, \qquad (12)\\ t_2 &= \gamma \bar{t}_2. \qquad (13) \end{aligned}$$

Note that, in time $\bar{T}$, the F frame will move a further distance of $v(\bar{t}_2 - \bar{t}_1)$. The second crest has to move a distance of x_2 (relative to observer) to reach the observer. And the corresponding time taken is

$$\mu_2 = |x_2|/v_{wo} \ . \qquad (14)$$

Equation (20) is our required master equation from which we derive all the familiar Doppler effect expressions.

So, for the observer, the second crest will reach at time

$$t'_2 = t_2 + \mu_2 \ . \tag{15}$$

Now, let us calculate the time period T of the wave with respect to the observer. The expression is

$$T = t'_2 - t'_1 \ . \tag{16}$$

Substituting for t'_2 and t'_1 using (15), (11), (14) and (10), we get

$$T = t_2 + \frac{|x_2|}{v_{wo}} - t_1 - \frac{|x_1|}{v_{wo}} \ . \tag{17}$$

Substituting for x_2, t_2 and x_1, t_1, and rearranging gives,

$$T = \gamma(\bar{t}_2 - \bar{t}_1)\left(1 + \frac{v}{v_{wo}}\right) \ . \tag{18}$$

Since $\bar{t}_2 - \bar{t}_1 = \bar{T}$, we get

$$T = \gamma\bar{T}\left(1 + \frac{v}{v_{wo}}\right) \ . \tag{19}$$

In terms of frequencies,

$$f = \bar{f}\frac{1}{\gamma\left(1 + \frac{v}{v_{wo}}\right)} \ . \tag{20}$$

Equation (20) is our required master equation from which we derive all the familiar Doppler effect expressions. Here, f is the shifted frequency and $\bar{f}$ is the original frequency. The results for the case when the source approaches the observer can be obtained by repeating the arguments above for negative values of the *time coordinates*, and reversing the direction of v, so that the wave velocity continues to be in the $+x$ direction. Hence, the result is the same as what we would get by replacing v by $-v$ in (20), i.e.,

$$f = \bar{f}\frac{1}{\gamma\left(1 - \frac{v}{v_{wo}}\right)} \ . \tag{21}$$

We consider v as a positive quantity always.

To calculate v_{wo}, we need to use Einstein's velocity addition formula.

4. Applying the Master Equation to Different Cases

Before considering the relativistic and non-relativistic situations, it is very important to understand the meaning of the effective wave velocity v_{wo}. We can accommodate the effect of motion of the medium, through which the wave propagates, in v_{wo}. To calculate v_{wo}, we need to use Einstein's velocity addition formula. For example, if v_{wm} is the velocity of the wave relative to the medium and v_{mo} is that of the medium relative to the observer, we can calculate,

$$v_{wo} = v_{wm} \oplus v_{mo} \ , \tag{22}$$

where $\oplus$ means relativistic addition or subtraction depending on whether the two velocities are in the same or opposite directions. Using this v_{wo} in (20) or (21) gives the correct shifted frequency. For the rest of this article, we focus on the case when the source approaches the observer, when (21) is the equation to be used.

Rearranging (21) yields

$$f = \bar{f}\frac{v_{wo}}{\gamma(v_{wo} - v)} \ . \tag{23}$$

Writing the relative velocity between the source and the observer as

$$v = v_{sm} \oplus v_{mo} \ , \tag{24}$$

and using (22), we can rewrite (23) as

$$f = \bar{f}\frac{1}{\gamma}\frac{v_{wm} \oplus v_{mo}}{(v_{wm} \oplus v_{mo} - (v_{sm} \oplus v_{mo}))} \ .$$

Here, v_{mo} is the velocity of the medium relative to the observer and v_{sm} is the velocity of the source relative to the medium. Also note that $v_{ab} = -v_{ba}$.

Equation (25) is nothing but our classical equation for the Doppler effect for 'sound', mentioned in (1).

4.1 *The Non-Relativistic Case*

In the non-relativistic case, Einstein's velocity addition formula reduces to the simple addition of velocities and we can set $\gamma = 1$. So we end up with

$$f = \bar{f}\frac{v_{wm} + v_{mo}}{v_{wm} - v_{sm}} \ . \tag{25}$$

Equation (25) is nothing but our classical equation for the Doppler effect for 'sound', mentioned in (1).

4.2 *The Relativistic Case*

Here we have two cases: (a) EM waves in vacuum, and (b) EM waves in any medium.

4.2.1 *EM Waves in Vacuum*

In this case, v_{wo} becomes c, the velocity of light in vacuum, and after rearrangement, (21) reads as

$$f = \bar{f}\frac{c}{\gamma(c - v)} \ , \tag{26}$$

and substituting the Lorentz factor γ in (26) yields

$$f = \bar{f}\sqrt{\frac{1 + v/c}{1 - v/c}}, \tag{27}$$

which is exactly (2).

4.2.2 *EM Waves in Any Medium*

In this case, (22) and (24) should be substituted in (21) and rearrangement yields

$$f = \bar{f}\frac{v_{wm} \oplus v_{mo}}{\gamma(v_{wm} \oplus v_{mo} - (v_{sm} \oplus v_{mo}))}. \tag{28}$$

This is the expression for the Doppler shifted frequency for this case. Since we need to do Einstein's velocity addition formula before further reduction, it is good to leave this equation as such. Here we accommodate the motion of the medium using the term v_{mo}.

Conclusion

Here, we derived a unified master expression for the Doppler effect for any wave in any moving or stationary medium and reduced it to familiar equations in cases of relativistic and non-relativistic situations. We also presented a relation (28) which gives the Doppler shifted frequency in case of EM waves in a medium moving relative to the observer.

Suggested Reading

[1] **O P Gupta, The Doppler effect: a unified approach for sound and light waves, *Phys. Educ.*, Vol.31, No.6, pp.351–355, 1996.**