Counting Your Way to the Sum of Squares Formula

Shailesh A Shirali

Shailesh Shirali is Director of Sahyadri School (KFI), Pune, and also Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been for long in the field of mathematics education, and has been closely involved with the Math Olympiad movement in the country. He is the author of many mathematics books addressed to high school students, and serves as an editor for *At Right Angles*.

Keywords
Combinatorial proof, algebraic proof, binomial coefficient, recursive relation, ordered pair, triple, chessboard, square, cube, fourth power, enumeration.

In this article, we use combinatorial reasoning to derive a formula for the sum of the first n squares:

$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

Following this we do the same for the formula for the sum of the first n cubes. Then we look at the problem more generally.

1. Combinatorial Reasoning

It is always a pleasure to find a proof for some known identity in a non-standard way, and one of the most pleasing varieties of proof is a *combinatorial proof.* The term 'combinatorial proof' is difficult to define precisely, but loosely speaking, it means a proof where we count the elements of a suitably chosen set in two different ways and then equate the two resulting expressions. In the two-part *Resonance* article [1], many such examples were studied. In this article, which may be regarded as a continuation of that one, we do the same for the formulas for the sum of the squares and the sum of the cubes of the first n natural numbers. Then we look for extensions of this reasoning.

In [2] (also quoted in [3]), Stanley makes the general comment about combinatorial proofs that not only are they generally more elegant and pleasing than algebraic proofs, they also frequently provide greater insight into 'why' a given identity is true.

2. Counting Triples in Two Ways

Let S_n denote the set $\{1, 2, 3, \ldots, n, n+1\}$. Let T_n denote the set of all triples (a, b, c) of elements from S_n such that $a < c$ and $b < c$. Note that $a = b$ is permitted in the definition. So:

$$T_n = \{(a,b,c) \ : \ a,b,c \in S_n, \ a < c, \ b < c\}. \qquad (1)$$

Example: $S_2 = \{1,2,3\}$,

$$T_2 = \big\{(1,1,3), \ (1,2,3), \ (2,1,3), \ (2,2,3), (1,1,2)\big\}.$$

By combining the two ways, we obtain a new formula for a known sum.

We wish to count the number of elements in T_n. We shall accomplish the count in two different ways. By combining the two ways we get a new, unfamiliar formula for a well-known sum.

Method–I

Select the element c first and then choose a and b, keeping in mind the conditions $a < c$ and $b < c$. These conditions imply that $c \geq 2$. Now observe the following:

(i) If $c = n+1$, there are n choices for both a and b, so the number of possible ordered pairs (a,b) is $n \times n = n^2$.

(ii) If $c = n$, there are $n-1$ choices for both a and b, so the number of possible ordered pairs (a,b) is $(n-1)^2$.

(iii) If $c = n-1$, there are $n-2$ choices for both a and b, so the number of possible ordered pairs (a,b) is $(n-2)^2$. And so on, down to:

(iv) If $c = 2$, the number of possible ordered pairs (a,b) is 1^2.

We see from this that the number of triples (a,b,c) of the required type is

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2.$$

A small tweak gives us a second formula for free.

Method–II

There are two kinds of triples satisfying the stated condition: those for which $a \neq b$, and those for which $a = b$. We count the two types separately.

- In triples of the first kind, the three numbers are distinct and hence can be chosen in $\binom{n+1}{3}$ ways. Each such choice yields two triples of the prescribed type (for the numbers in the first two places can be swapped, there being no order relation between them). Hence the number of triples of this kind is $2 \cdot \binom{n+1}{3}$.
- Triples of the second kind have the form (a, a, c) where $a < c$, hence their number is equal to the number of two-element subsets of S_n; this number is $\binom{n+1}{2}$.

It follows readily from the above that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = 2 \cdot \binom{n+1}{3} + \binom{n+1}{2}. \quad (2)$$

This gives us a brand new formula for the sum of the squares of the first n positive integers!

A small tweak gives us a second formula, for free! For, we have the following identity for the binomial coefficients which comes from the well known recursive relation which the binomial coefficients satisfy:

$$\binom{n+1}{3} + \binom{n+1}{2} = \binom{n+2}{3}.$$

Hence we have, from (2):

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \binom{n+2}{3} + \binom{n+1}{3}. \quad (3)$$

This is our second formula. Here is a sample computation that verifies the formula for $n = 5$. We have:

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$, and

$$\binom{7}{3} + \binom{6}{3} = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} + \frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 35 + 20 = 55.$$

Note that we have derived (3) algebraically from (2). It is a nice challenge to find a direct combinatorial proof of result (3). Later in this article we do just this.

Deriving the well-known formula

Using the above result and known formulas for the combinatorial coefficients, we now derive the familiar formula for the sum $\sum_{r=1}^{n} r^2$:

$$\begin{aligned} \sum_{r=1}^{n} r^2 &= \binom{n+2}{3} + \binom{n+1}{3} \\ &= \frac{n(n+1)}{6} \left[(n+2) + (n-1)\right] \\ &= \frac{n(n+1)(2n+1)}{6}. \end{aligned} \tag{4}$$

3. Another Route to the Same Formula

Here is an entirely different way of arriving at the formula; again, using counting.

Lemma 1. *For any positive integer n, we have:*

$$n^2 = 2 \cdot \binom{n}{2} + \binom{n}{1}. \tag{5}$$

Proof. Let $S = \{1, 2, 3, \ldots, n-1, n\}$, and consider all ordered pairs (a, b) such that $a, b \in S$. There are clearly n^2 such pairs.

Now let's count them another way. The number of ordered pairs (a, b) where $a \neq b$ is twice the number of two-element subsets of S; hence it is equal to $2 \cdot \binom{n}{2}$.

Let us count according to the largest number in the set.

The number of ordered pairs (a, b) where $a = b$ is equal to the number of elements of S, which is $\binom{n}{1}$. Hence the claim. □

Lemma 2. *For any positive integer n, we have:*

$$\binom{n+1}{3} = \sum_{k=2}^{n} \binom{k}{2}. \tag{6}$$

Proof. We count the number of three-element subsets of the set

$$\{1, 2, 3, \ldots, n, n+1\}.$$

By definition the number is $\binom{n+1}{3}$. But let us count according to the largest number in the set, which cannot be less than 3:

- The number of three-element subsets with largest element $n+1$ is $\binom{n}{2}$.
- The number of three-element subsets with largest element n is $\binom{n-1}{2}$.
- The number of three-element subsets with largest element $n-1$ is $\binom{n-2}{2}$. And so on, down to:
- The number of three-element subsets with largest element 3 is $\binom{2}{2}$.

It follows that $\binom{n+1}{3} = \binom{n}{2} + \binom{n-1}{2} + \binom{n-2}{2} + \cdots + \binom{3}{2} + \binom{2}{2}$. Using the same reasoning (but for two-element subsets) we find that:

$$\binom{n+1}{2} = \sum_{k=1}^{n} \binom{k}{1}. \tag{7}$$

(Note incidentally that this yields another proof of the well-known identity for the sum of the first n natural numbers.) We now use Lemmas 1 and 2 in succession to get:

$$n^2 + (n-1)^2 + \cdots + 2^2 + 1^2$$
$$= \sum_{k=1}^{n} \left[2 \cdot \binom{k}{2} + \binom{k}{1} \right]$$
$$= 2 \cdot \binom{n+1}{3} + \binom{n+1}{2}.$$

This yields another proof for the formula for the sum of the first n natural numbers.

4. Squares on a Chessboard

In passing, we note that the expression for the sum of the squares of the first n natural numbers arises in connection with the following chessboard problem:

> *On a 8×8 chessboard, how many squares are there, of all possible sizes?*

The squares here are assumed to be formed by the grid lines of the chessboard. (See *Figure* 1. Some candidates squares have been shown using a thick red border.)

It is easy to see that the number of squares of size 1×1 is 8^2; the number of squares of size 2×2 is 7^2; the number of squares of size 3×3 is 6^2; and so on, down to: the number of squares of size 8×8 is 1^2. Hence the total number of squares formed by the grid lines of the board is:

$$1^2 + 2^2 + 3^2 + \cdots + 8^2 = 204.$$

In the same way, we see that for a chessboard of size $n \times n$, the total number of squares formed by the grid lines of the board is:

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2.$$

You may wonder whether the chessboard interpretation allows for another combinatorial derivation of the formula for the sum of the squares of the first n natural numbers. The answer: Yes indeed. Here's how we do it.

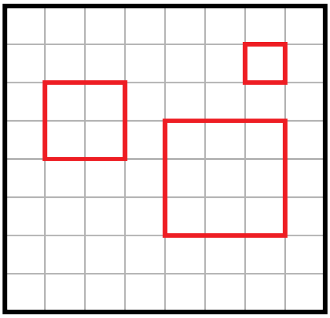


Figure 1.

Draw diagonal grid lines as shown in *Figure* 2, running from the south-west (SW) direction to the north-east (NE) direction. (The figure has been shown for the case $n = 8$.) At the SW corner of each such diagonal line, we note the number of lattice points it has (a lattice point being formed by the intersection of two grid lines). The numbers go thus: 2, 3, 4, 5, 6, 7, 8, 9, 8, 7, 6, 5, 4, 3, 2.

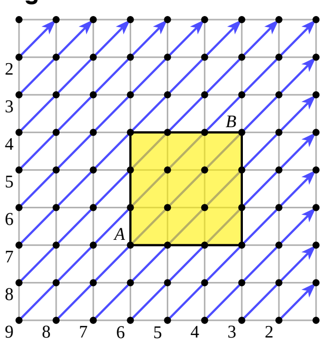
Now we make a crucial observation, exploiting the geometry of a square: *If we are given a diagonal of a square, we can construct the square.* That is, the diagonal uniquely fixes the square. For example, in *Figure* 2, the yellow square is fixed by diagonal AB. It follows that there is a one-to-one correspondence between the squares formed by the grid lines and pairs of points on these various diagonal lines. Using the numbers noted at the SW corners of the diagonal lines, we deduce that the total number of squares is equal to the following number:

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{8}{2} + \binom{9}{2}$$
$$+ \binom{8}{2} + \binom{7}{2} + \cdots + \binom{3}{2} + \binom{2}{2}.$$

Using Lemma 2, this reduces to:

$$\binom{10}{3} + \binom{9}{3} = 120 + 84 = 204.$$

Figure 2.



For the general case (i.e., a $n \times n$ chessboard), the total number of squares formed by the grid lines is equal to:

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2} + \binom{n+1}{2}$$
$$+ \binom{n}{2} + \binom{n-1}{2} + \cdots + \binom{3}{2} + \binom{2}{2},$$

and this is equal to

$$\binom{n+2}{3} + \binom{n+1}{3}.$$

Hence we have:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \binom{n+2}{3} + \binom{n+1}{3}.$$

5. Reaching Further: Sums of Cubes

This reasoning may easily be extended. How many triples (a, b, c) are there for which $a, b, c \in \{1, 2, \ldots, n\}$? Clearly the number is n^3. These triples are of three kinds, and we count each type separately:

- The number of triples in which all three elements are distinct is $3! \cdot \binom{n}{3} = 6 \cdot \binom{n}{3}$.
- The number of triples with two distinct elements (i.e., with one element repeated) is $3{\cdot}2{\cdot}\binom{n}{2} = 6{\cdot}\binom{n}{2}$.
- The number of triples with a single element repeated thrice is $\binom{n}{1}$.

Hence:

$$n^3 = 6 \cdot \binom{n}{3} + 6 \cdot \binom{n}{2} + \binom{n}{1}. \tag{8}$$

Using exactly the same reasoning as used above, we get a new formula for the sum of the cubes of the first n positive integers:

$$\sum_{k=1}^{n} k^3 = 6 \cdot \binom{n+1}{4} + 6 \cdot \binom{n+1}{3} + \binom{n+1}{2}. \tag{9}$$

Another way of expressing this is the following:

$$\sum_{k=1}^{n} k^3 = 6 \cdot \binom{n+2}{4} + \binom{n+1}{2}. \tag{10}$$

We may check algebraically that the expression on the right side is equal to $\frac{1}{4}n^2(n+1)^2$.

6. And Further Still . . . : Sums of Fourth Powers

Will this reasoning work for the sum $\sum_{k=1}^{n} k^4$? Yes. Now we count 4-tuples (a, b, c, d) where $a, b, c, d \in \{1, 2, \ldots, n\}$. Counting them in two different ways we arrive at the following identity (the coefficients have to be worked out very carefully; it is easy to go wrong):

$$\begin{aligned} n^4 &= 4! \cdot \binom{n}{4} + \frac{3 \times 4!}{2!} \cdot \binom{n}{3} \\ &\quad + \frac{2 \times 4!}{3!} \cdot \binom{n}{2} + \frac{4!}{2! \times 2!} \cdot \binom{n}{2} + \binom{n}{1} \\ &= 24 \cdot \binom{n}{4} + 36 \cdot \binom{n}{3} + 8 \cdot \binom{n}{2} + 6 \cdot \binom{n}{2} + \binom{n}{1}, \end{aligned} \tag{11}$$

from which it follows that:

$$\begin{aligned} \sum_{k=1}^{n} k^4 &= 24 \cdot \binom{n+1}{5} + 36 \cdot \binom{n+1}{4} \\ &\quad + 14 \cdot \binom{n+1}{3} + \binom{n+1}{2}. \end{aligned} \tag{12}$$

Here's a numerical check for the sample value $n = 5$. The sum of the fourth powers is $1+16+81+256+625 = 979$. The formula gives: $24 \times 6 + 36 \times 15 + 14 \times 20 + 15 = 979$.

7. The General Case

Let us now extend this reasoning to the general case, i.e., to sums of kth powers. While doing so we see some known algebraic facts in a new light.

Counting the k-tuples of elements of $\{1, 2, \ldots, n-1, n\}$ in two ways reveals that for each positive integer k, positive integers $a_{k,k}$, $a_{k,k-1}$, $\ldots$, $a_{k,1}$ can be found such that

$$n^k = \sum_{i=1}^{k} \binom{n}{i} \cdot a_{k,i}. \tag{13}$$

The k-tuples can also be thought of as k-letter words, the letters being drawn from a base alphabet containing n letters. The total number of possible words is n^k. If we subdivide the set of possible words into subsets based on how many distinct letters are used in the word, we get the above relation. Note that $\binom{n}{j} \cdot a_{k,j}$ is the number of different k-letter words in which i distinct letters are used, for $j = 1, 2, \ldots, k$. The coefficients $a_{k,j}$ can be explicitly computed (note that $a_{k,j}$ is the number of possible k-letter words using j **given** letters, with the requirement that every letter must be 'used'):

$$\begin{aligned} a_{k,1} &= 1, \\ a_{k,2} &= 2^k - 2, \\ a_{k,3} &= 3^k - 3 \cdot 2^k + 3, \\ a_{k,4} &= 4^k - 4 \cdot 3^k + 12 \cdot 2^k - 4, \quad \ldots \\ a_{k,k-1} &= \frac{k!(k-1)}{2}, \\ a_{k,k} &= k!. \end{aligned}$$

The above expressions were obtained using combinatorial reasoning, but they can also be obtained (perhaps more simply) using algebraic reasoning, by substituting in turn $n = 1,\ 2,\ \ldots,\ k-1,\ k$ in (2) and proceeding recursively:

- $n = 1$ yields $a_{k,1} = 1$;
- $n = 2$ yields $a_{k,2} = 2^k - 2$;
- $n = 3$ yields $a_{k,3} = 3^k - 3 \cdot 2^k + 3$;

and so on. Now, having obtained these coefficients, we use the relation

$$n^k = \sum_{j=1}^{k} a_{k,j} \binom{n}{j}$$

If we subdivide the set of possible words into subsets based on how many distinct letters are used in the word, we get the above relation.

and invoke the reasoning used in Sections 2, 3, 4; we get:

$$\sum_{r=1}^{n} r^k = \sum_{j=1}^{k} a_{k,j} \binom{n+1}{j+1}. \tag{14}$$

8. A Striking Algebraic Pattern

It is apparent from the last relation that for each positive integer k, the function $P_k(n) := \sum_{r=1}^{n} r^k$ is a polynomial in n with degree $k+1$.

But more can be said, from the fact that the successive terms in (14) are, respectively, multiples of:

$$(n+1)n, \;\; (n+1)n(n-1), \;\; (n+1)n(n-1)(n-2), \;\; \ldots.$$

We thus obtain the following non-trivial result:

Theorem 1. *For each positive integer k, the polynomial $P_k(n)$ has $n(n+1)$ as a factor.*

A second such result can be obtained. It is easy to see that the leading coefficient in $P_k(n)$, i.e., the coefficient of n^{k+1}, is $\frac{1}{k+1}$. What about the next one, i.e., the coefficient of n^k? The answer comes as a surprise: it turns out to be the same for all k. We state and prove this result below. Note however that the reasoning used is algebraic, not combinatorial.

Theorem 2. *For each positive integer k, the coefficient of n^k in $P_k(n)$ is $\frac{1}{2}$.*

Proof. In (2), only the first two terms contribute to the coefficient of n^k, namely $\binom{n+1}{k+1}$ and $\binom{n+1}{k}$. We examine these terms in turn. For the first term:

$$\binom{n+1}{k+1} = \frac{(n+1)n(n-1)(n-2)\cdots(n-k+1)}{(k+1)!}.$$

The coefficient of n^k in this is

$$\frac{1+0-(1+2+\cdots+k-1)}{(k+1)!} = -\frac{k-2}{2\cdot k!}.$$

The coefficient of n^k in $\binom{n+1}{k}$ is $\frac{1}{k!}$, as n^k is the leading term. Therefore, the coefficient of n^k in $P_k(n)$ is

$$\begin{aligned} & -a_{k,k}\cdot\left(\frac{k-2}{2\cdot k!}\right)+a_{k,k-1}\cdot\frac{1}{k!} \\ = & -k!\cdot\left(\frac{k-2}{2\cdot k!}\right)+\frac{k!(k-1)}{2}\cdot\frac{1}{k!} \\ = & -\frac{k-2}{2}+\frac{k-1}{2} = \frac{1}{2}. \end{aligned}$$

A remarkable result!

Remark

The fact that the coefficients of n^{k+1} and n^k in $P_k(n)$ are $\frac{1}{k+1}$ and $\frac{1}{2}$ respectively may be captured in a single statement: For each positive integer k,

$$\lim_{n\to\infty}\frac{P_k(n)-n^{k+1}/(k+1)}{n^k} = \frac{1}{2}. \qquad (15)$$

List of the polynomials

Here is a list of the polynomials $P_k(n)$ for $k=1,2,3,4,5$:

$$P_1(n)=\frac{1}{2}n^2+\frac{1}{2}n,$$

$$P_2(n)=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n,$$

$$P_3(n)=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2,$$

$$P_4(n)=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n,$$

$$P_5(n)=\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2.$$

There are lots of patterns to be found in this algebraic maze.

There are lots of patterns to be found in this algebraic maze (a good many of which are visible to the eye), but it is possible that they are not accessible using purely combinatorial arguments. But it is certainly a lot of fun and highly instructive to see how far the combinatorial approach can take us.

Suggested Reading

[1] Shailesh A Shirali, Combinatorial Proofs and Algebraic Proofs (in two parts), *Resonance,* (pub: Indian Academy of Sciences), July 2013 & August 2013 (available at verb+ http://www.ias.ac.in/resonance/Volumes/18/07/0630-0645.pdf and http://www.ias.ac.in/resonance/Volumes/18/08/0738-0747.pdf respectively)

[2] Richard P Stanley, *Enumerative Combinatorics*, Volume I, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, pp.11–12, 1997, ISBN 0-521-55309-1. Available as a free download, courtesy of the author: http://www-math.mit.edu/~rstan/ec/ec1.pdf

[3] https://en.wikipedia.org/wiki/Combinatorial_proof

Address for Correspondence
Shailesh A Shirali
Sahyadri School
Tiwai Hill, Rajgurunagar
Pune 410 513, India.
Email:
shailesh.shirali@gmail.com