Sums of Generalized Harmonic Series

For Kids from Five to Fifteen

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We examine the remarkable connection, first discovered by Beukers, Kolk and Calabi, between $\zeta(2n)$, the value of the Riemann Zeta-function at an even positive integer, and the volume of some $2n$-dimensional polytope. It can be shown that this volume is equal to the trace of a compact self-adjoint operator. We provide an explicit expression for the kernel of this operator in terms of Euler polynomials. This explicit expression makes it easy to calculate the volume of the polytope and hence $\zeta(2n)$. In the case of odd positive integers, we rediscover an integral representation for $\zeta(2n+1)$, obtained by a different method by Cvijović and Klinowski. Finally, we indicate that the origin of the miraculous Beukers–Kolk–Calabi change of variables in the multidimensional integral, which is at the heart of this circle of ideas, can be traced to the amoeba associated with the certain Laurent polynomial. The article is dedicated to the memory of Vladimir Arnold (1937–2010).

1. Introduction

In a nice little book [1], Vladimir Arnold has collected 77 mathematical problems for kids from 5 to 15 to stimulate critical thinking in them. Problem 51 in this book asks the reader to calculate the sum of the inverse squares and prove Euler's celebrated formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{1}$$

Well, there are many ways to do this (see, for example,

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[2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein), some maybe even accessible for kids under fifteen[1]. However, in this note we concentrate on the approach of Beukers, Kolk and Calabi [11], further elaborated by Elkies in [12]. This approach incorporates pleasant features which all the kids (and even some adults) adore: simplicity, magic and the depth that allows one to go beyond the particular case (1). The simplicity, however, is not everywhere explicit in [11] and [12], while the magic longs for explanation after the first admiration fades away. Below we will try to enhance the simplicity of the approach and somewhat uncover the secret of the magic.

[1] Editor's note: This perhaps applies to Russian kids!

The article is organized as follows. In the first two sections, we reconsider the evaluation of $\zeta(2)$ and $\zeta(3)$ so that technical details of the general case do not obscure the simple underlying ideas. Then, we elaborate the general case and give the main result of this work, the formula for the kernel which allows us to simplify considerably the evaluation of $\zeta(2n)$ from [11, 12] and re-derive Cvijović and Klinowski's integral representation [13] for $\zeta(2n+1)$. Finally, we ponder over the mysterious relations between the sums of generalized harmonic series and amoebas, first indicated by Passare in [10]. This relation enables us to uncover somewhat the origin of the Beukers–Kolk–Calabi's highly non-trivial change of variables.

The approach of Beukers, Kolk and Calabi has the features of simplicity, magic and depth. But, the simplicity is not explicit and the magic longs for explanation after the first admiration fades away.

Evaluation of $\zeta(2)$

Recall the definition of the Riemann Zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2}$$

The sum (1) is just $\zeta(2)$ which we will now evaluate following the method of Beukers, Kolk and Calabi [11].

This change of variables transforms the unit square to an isosceles triangle.

Our starting point will be the dilogarithm function

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}. \qquad (3)$$

Clearly, $\mathrm{Li}_2(0) = 0$ and $\mathrm{Li}_2(1) = \zeta(2)$. Differentiating (3), we get

$$x\,\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x),$$

and, therefore,

$$\zeta(2) = \mathrm{Li}_2(1) = -\int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x = \iint_{\Box} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}, \qquad (4)$$

where $\Box = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$ is the unit square. Let us note

$$\iint_{\Box} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} + \iint_{\Box} \frac{\mathrm{d}x\,\mathrm{d}y}{1+xy} = 2\iint_{\Box} \frac{\mathrm{d}x\,\mathrm{d}y}{1-x^2y^2}, \qquad (5)$$

and

$$\iint_{\Box} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} - \iint_{\Box} \frac{\mathrm{d}x\,\mathrm{d}y}{1+xy} = \frac{1}{2}\iint_{\Box} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}, \qquad (6)$$

where the last equation follows from

$$\iint_{\Box} \frac{2xy}{1-x^2y^2}\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\iint_{\Box} \frac{\mathrm{d}(x^2)\,\mathrm{d}(y^2)}{1-x^2y^2} = \frac{1}{2}\iint_{\Box} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}.$$

It follows from equations (5) and (6) that

$$\zeta(2) = \frac{4}{3}\iint_{\Box} \frac{\mathrm{d}x\,\mathrm{d}y}{1-x^2y^2}. \qquad (7)$$

Now let us make the magic Beukers–Kolk–Calabi change of variables in this two-dimensional integral [11]

$$x = \frac{\sin u}{\cos v}, \quad y = \frac{\sin v}{\cos u}, \qquad (8)$$

with Jacobian determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\cos u}{\cos v} & \frac{\sin v \sin u}{\cos^2 u} \\ \frac{\sin u \sin v}{\cos^2 v} & \frac{\cos v}{\cos u} \end{vmatrix} = 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 v \cos^2 u} = 1 - x^2 y^2.$$

Then miraculously

$$\zeta(2) = \frac{4}{3} \iint_{\Delta} \mathrm{d}u\, \mathrm{d}v = \frac{4}{3}\,\mathrm{Area}(\Delta), \tag{9}$$

Editor's note: For a change of variables, the integral gets scaled by the Jacobian determinant.

where Δ is the image of the unit square $\square$ under the transformation $(x,y) \to (u,v)$. It is easy to show that Δ is the isosceles right triangle $\Delta = \{(u,v) : u \geq 0, v \geq 0, u+v \leq \pi/2\}$ and, therefore,

$$\zeta(2) = \frac{4}{3}\,\frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{6}. \tag{10}$$

"Beautiful – even more so, as the same method of proof extends to the computation of $\zeta(2k)$ in terms of a $2k$-dimensional integral, for all $k \geq 1$" [14]. However, before considering the general case, we check whether the trick works for $\zeta(3)$.

3. Evaluation of $\zeta(3)$

In the case of $\zeta(3)$, we begin with trilogarithm

$$\mathrm{Li}_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}, \tag{11}$$

and using

$$x\,\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}_3(x) = \mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-y)}{y}\mathrm{d}y,$$

we get

$$\zeta(3) = \mathrm{Li}_3(1) = -\int_0^1 \frac{\mathrm{d}x}{x} \int_0^x \frac{\ln(1-y)}{y}\mathrm{d}y. \tag{12}$$

A change of variables of the unit cube similar to that of the square unfortunately leads to a complicated integral. A hyperbolic version gives an interesting integral which too is not possible to evaluate; hence zeta(3) is more complicated than zeta(2).

But

$$-\frac{1}{x}\int_0^x \frac{\ln(1-y)}{y}\mathrm{d}y = -\int_0^1 \frac{\ln(1-xz)}{xz}\mathrm{d}z$$
$$= \int_0^1 \mathrm{d}z \int_0^1 \frac{\mathrm{d}y}{1-xyz},$$

and finally

$$\zeta(3) = \mathrm{Li}_3(1) = \iiint_{\square_3} \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-xyz}, \tag{13}$$

where $\square_3 = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ is the unit cube. By a similar trick as before, we can transform (13) into the integral

$$\zeta(3) = \frac{8}{7}\iiint_{\square_3} \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-x^2y^2z^2}, \tag{14}$$

and here the analogy with the previous case ends, unfortunately, because the generalization of the Beukers–Kolk–Calabi change of variables does not lead in this case to a simple integral. However, it is interesting to note that the hyperbolic version of this change of variables

$$x = \frac{\sinh u}{\cosh v}, \quad y = \frac{\sinh v}{\cosh w}, \quad z = \frac{\sinh w}{\cosh u} \tag{15}$$

does indeed produce an interesting result

$$\zeta(3) = \frac{8}{7}\iiint_{U_3} \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = \frac{8}{7}\mathrm{Vol}(\mathrm{U}_3), \tag{16}$$

where U_3 is a complicated 3-dimensional shape defined by the inequalities

$$u \ge 0, \quad v \ge 0, \quad w \ge 0, \quad \sinh u \le \cosh v, \quad \sinh v \le \cosh w,$$
$$\sinh w \le \cosh u.$$

Unfortunately, there is no obvious simple way to calculate the volume of U_3.

However, there is a second way to convert the integral (12) for $\zeta(3)$ in which the Beukers–Kolk–Calabi change of variables still plays a helpful role. We begin with the identity

$$\zeta(3) = -\int_0^1 \frac{\mathrm{d}x}{x} \int_0^x \frac{\ln(1-y)}{y}\mathrm{d}y = -\iint_D \frac{\ln(1-y)}{xy}\,\mathrm{d}x\,\mathrm{d}y, \tag{17}$$

where the domain of the 2-dimensional integration is the triangle $D = \{(x,y) : x \geq 0, y \geq 0, y \leq x\}$. Interchanging the order of integration in (17), we get

$$\zeta(3) = -\int_0^1 \frac{\ln(1-y)}{y}\,\mathrm{d}y \int_y^1 \frac{\mathrm{d}x}{x},$$

which can be transformed further as follows

$$\zeta(3) = \int_0^1 \frac{\ln(1-y)\ln y}{y}\,\mathrm{d}y = -\int_0^1 \ln y\,\mathrm{d}y \int_0^1 \frac{\mathrm{d}x}{1-xy}$$
$$= -\iint_\square \frac{\ln y}{1-xy}\,\mathrm{d}x\,\mathrm{d}y,$$

or in a more symmetrical form

$$\zeta(3) = -\frac{1}{2}\iint_\square \frac{\ln(xy)}{1-xy}\,\mathrm{d}x\,\mathrm{d}y. \tag{18}$$

Note that

$$\iint_\square \frac{2xy\,\ln(xy)}{1-x^2y^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\iint_\square \frac{\ln(x^2y^2)}{1-x^2y^2}\,\mathrm{d}(x^2)\,\mathrm{d}(y^2)$$
$$= \frac{1}{4}\iint_\square \frac{\ln(xy)}{1-xy}\,\mathrm{d}x\,\mathrm{d}y.$$

Therefore, we can modify (5) and (6) accordingly and using them transform (18) into

$$\zeta(3) = -\frac{4}{7} \iint\limits_{\square} \frac{\ln(xy)}{1 - x^2y^2}\, \mathrm{d}x\, \mathrm{d}y. \tag{19}$$

At this point we can use the Beukers–Kolk–Calabi change of variables (8) in (19) and as a result we get

$$\begin{aligned} \zeta(3) &= -\frac{4}{7} \iint\limits_{\Delta} \ln(\tan u \, \tan v)\, \mathrm{d}u\, \mathrm{d}v \\ &= -\frac{8}{7} \iint\limits_{\Delta} \ln(\tan u)\, \mathrm{d}u\, \mathrm{d}v. \end{aligned} \tag{20}$$

But this equation indicates that

$$\begin{aligned} \zeta(3) &= -\frac{8}{7} \int\limits_0^{\pi/2} \mathrm{d}u \, \ln(\tan u) \int\limits_0^{\pi/2-u} \mathrm{d}v \\ &= -\frac{8}{7} \int\limits_0^{\pi/2} \left(\frac{\pi}{2} - u\right) \ln(\tan u)\, \mathrm{d}u, \end{aligned}$$

which after the substitution $x = \frac{\pi}{2} - u$ becomes

$$\zeta(3) = -\frac{8}{7} \int\limits_0^{\pi/2} x \ln(\cot x)\, \mathrm{d}x = \frac{8}{7} \int\limits_0^{\pi/2} x \ln(\tan x)\, \mathrm{d}x. \tag{21}$$

But

$$\begin{aligned} \int\limits_0^{\pi/2} \ln(\tan x)\, \mathrm{d}x &= -\int\limits_{\pi/2}^{0} \ln(\cot u)\, \mathrm{d}u \\ &= -\int\limits_0^{\pi/2} \ln(\tan u)\, \mathrm{d}u = 0, \end{aligned}$$

which allows us to rewrite (21) as follows

$$\zeta(3) = \frac{8}{7}\int_0^{\pi/2} \left(x - \frac{\pi}{4}\right) \ln(\tan x)\,\mathrm{d}x$$

$$= \frac{8}{7}\int_0^{\pi/2} \ln(\tan x)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2} - \frac{\pi}{4}x\right)\,\mathrm{d}x,$$

and after integration by parts and rescaling $x \to x/2$ we end with

$$\zeta(3) = \frac{1}{7}\int_0^{\pi} \frac{x(\pi - x)}{\sin x}\,\mathrm{d}x. \tag{22}$$

This is certainly an interesting result. Note that until quite recently very few definite integrals of this kind, involving cosecant or secant functions, were known and present in standard tables of integrals [15, 16, 17]. In fact (22) is a special case of the more general result [13] which we are going now to establish.

4. The General Case of $\zeta(2n)$

The evaluation of $\zeta(2)$ can be straightforwardly generalized. The polylogarithm function

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \tag{23}$$

obeys

$$x\,\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}_s(x) = \mathrm{Li}_{s-1}(x),$$

and hence

$$\mathrm{Li}_s(x) = \int_0^{x} \frac{\mathrm{Li}_{s-1}(y)}{y}\,\mathrm{d}y. \tag{24}$$

Repeated application of this identity allows to write

$$\zeta(n) = \mathrm{Li}_n(1) = \int\limits_0^1 \frac{\mathrm{d}x_1}{x_1} \int\limits_0^{x_1} \frac{\mathrm{d}x_2}{x_2} \cdots \int\limits_0^{x_{n-2}} \frac{\mathrm{d}x_{n-1}}{x_{n-1}} \left[-\ln\left(1-x_{n-1}\right)\right]. \quad (25)$$

After rescaling

$$x_1 = y_1, \quad x_2 = x_1 y_2, \quad x_3 = x_2 y_3, \ldots, x_{n-1} = x_{n-2} y_{n-1} = y_1 y_2 \cdots y_{n-1},$$

and using

$$\int\limits_0^1 \frac{\mathrm{d}y_n}{1 - y_1 y_2 \cdots y_n} = -\frac{1}{y_1 y_2 \cdots y_{n-1}} \ln\left(1 - y_1 y_2 \cdots y_{n-1}\right),$$

we get

$$\zeta(n) = \int \underset{\square_n}{\cdots} \int \frac{\mathrm{d}y_1\, \mathrm{d}y_2 \cdots \mathrm{d}y_n}{1 - y_1 y_2 \cdots y_n}, \quad (26)$$

where $\square_n$ is n-dimensional unit hypercube. The analogs of (5) and (6) are

$$\int \underset{\square_n}{\cdots} \int \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{1 - x_1 \cdots x_n} + \int \underset{\square_n}{\cdots} \int \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{1 + x_1 \cdots x_n} = 2 \int \underset{\square_n}{\cdots} \int \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{1 - x_1^2 \cdots x_n^2}$$

and

$$\int \underset{\square_n}{\cdots} \int \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{1 - x_1 \cdots x_n} - \int \underset{\square_n}{\cdots} \int \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{1 + x_1 \cdots x_n} = \frac{1}{2^{n-1}} \int \underset{\square_n}{\cdots} \int \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{1 - x_1 \cdots x_n},$$

from which it follows that (26) is equivalent to

$$\zeta(n) = \frac{2^n - 1}{2^n} \int_{\square_n} \cdots \int \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{1 - x_1^2 \cdots x_n^2}. \tag{27}$$

If we now make a change of variables that generalizes (8), namely

$$x_1 = \frac{\sin u_1}{\cos u_2}, \quad x_2 = \frac{\sin u_2}{\cos u_3}, \ldots, \; x_{n-1} = \frac{\sin u_{n-1}}{\cos u_n}, \quad x_n = \frac{\sin u_n}{\cos u_1}. \tag{28}$$

we, in general, encounter a problem because the Jacobian of (28) is [11, 12]

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)} = 1 - (-1)^n \, x_1^2 x_2^2 \cdots x_n^2,$$

and, therefore, only for even n we will get a 'simple' integral. For the hyperbolic version of (28),

$$x_1 = \frac{\sinh v_1}{\cosh v_2}, \quad x_2 = \frac{\sinh v_2}{\cosh v_3}, \ldots, x_{n-1} = \frac{\sinh v_{n-1}}{\cosh v_n}, \quad x_n = \frac{\sinh v_n}{\cosh v_1}, \tag{29}$$

the Jacobian has the 'right' form

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(v_1, \ldots, v_n)} = 1 - x_1^2 x_2^2 \cdots x_n^2,$$

and we get

$$\zeta(n) = \frac{2^n}{2^n - 1} \int_{U_n} \cdots \int dv_1 \cdots \mathrm{d}v_n = \frac{2^n}{2^n - 1} \, \mathrm{Vol}_n(U_n). \tag{30}$$

However, like U_3 in (16), the figure U_n, which is defined by the variables v_i under the change of variables given by (29), has a complicated shape and it is not altogether clear how to calculate its n-dimensional volume

$\mathrm{Vol}_n(\mathrm{U}_n)$ (nevertheless, a hyperbolic version can lead to some new insights [8, 9]). Therefore, for a moment, we concentrate on the even values of n for which (28) works perfectly well and leads to [11, 12]

$$\zeta(2n) = \frac{2^{2n}}{2^{2n}-1} \int_{\Delta_{2n}} \cdots \int \mathrm{d}u_1 \cdots \mathrm{d}u_n = \frac{2^{2n}}{2^{2n}-1}\, \mathrm{Vol}_{2n}(\Delta_{2n}), \tag{31}$$

where Δ_n is a n-dimensional polytope defined through the inequalities

$$\Delta_n = \left\{(u_1, \ldots, u_n) : u_i \geq 0,\ u_i + u_{i+1} \leq \frac{\pi}{2}\right\}. \tag{32}$$

It is assumed in (32) that u_i are indexed cyclically (mod n) and therefore $u_{n+1} = u_1$.

There exists an elegant method due to Elkies [12] for calculating the n-volume of Δ_n (earlier calculations of this type can be found in [18]). Obviously

$$\mathrm{Vol}_n(\Delta_n) = \left(\frac{\pi}{2}\right)^n \mathrm{Vol}_n(\delta_n), \tag{33}$$

where $\mathrm{Vol}_n(\delta_n)$ is the n-dimensional volume of the rescaled polytope

$$\delta_n = \{(u_1, \ldots, u_n) : u_i \geq 0,\ u_i + u_{i+1} \leq 1\}. \tag{34}$$

If we introduce the characteristic function[2] $K_1(u, v)$ of the isosceles right triangle $\{(u, v) : u, v \geq 0,\ u + v \leq 1\}$ that is 1 inside the triangle and 0 outside of it, then [12]

[2] This is the function which takes value 1 at points inside the triangle and 0 outside of it.

$$\mathrm{Vol}_n(\delta_n) = \int_0^1 \ldots \int_0^1 \prod_{i=1}^{n} K_1(u_i, u_{i+1})\, \mathrm{d}u_1 \ldots \mathrm{d}u_n = \int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2\, K_1(u_1, u_2) \ldots$$

$$\int_0^1 \mathrm{d}u_{n-1}\, K_1(u_{n-2}, u_{n-1}) \int_0^1 \mathrm{d}u_n\, K_1(u_{n-1}, u_n)\, K_1(u_n, u_1). \tag{35}$$

Let us note that $K_1(u, v)$ can be interpreted [12] as the kernel of the linear operator $\hat{T}$ on the Hilbert space $L^2(0, 1)$, defined as follows

$$(\hat{T}f)(u) = \int_0^1 K_1(u, v) f(v)\, \mathrm{d}v = \int_0^{1-u} f(v)\, \mathrm{d}v. \tag{36}$$

Then (35) shows that $\mathrm{Vol}_n(\delta_n)$ equals just to the trace of the operator $\hat{T}^n$:

$$\mathrm{Vol}_n(\delta_n) = \int_0^1 K_n(u_1, u_1)\, \mathrm{d}u_1, \tag{37}$$

whose kernel $K_n(u, v)$ obeys the recurrence relation

$$K_n(u, v) = \int_0^1 K_1(u, u_1)\, K_{n-1}(u_1, v)\, \mathrm{d}u_1. \tag{38}$$

Surprisingly, we can find a simple enough solution of this recurrence relation [19]. Namely,

$$K_{2n}(u, v) = (-1)^n \frac{2^{2n-2}}{(2n-1)!} \times \left\{ \left[E_{2n-1}\left(\frac{u+v}{2}\right) + E_{2n-1}\left(\frac{u-v}{2}\right) \right] \theta(u-v) + \left[E_{2n-1}\left(\frac{u+v}{2}\right) + E_{2n-1}\left(\frac{v-u}{2}\right) \right] \theta(v-u) \right\}, \tag{39}$$

Editor's note: A linear transformation such as one defined by equation (36) is called an integral operator and the function K is called its kernel. The integral defined by (37) is called its trace.

Editor's note: Many formulae such as (39) and (40) are easy to prove once they are stated; it is in their discovery that the magic lies. The magic here is revealed.

and

$$K_{2n+1}(u,v) = (-1)^n \frac{2^{2n-1}}{(2n)!} \times \left\{ \left[E_{2n}\left(\frac{1-u+v}{2}\right) + E_{2n}\left(\frac{1-u-v}{2}\right) \right] \theta(1-u-v) + \left[E_{2n}\left(\frac{1-u+v}{2}\right) - E_{2n}\left(\frac{u+v-1}{2}\right) \right] \theta(u+v-1) \right\}. \quad (40)$$

In these formulas $E_n(x)$ are the Euler polynomials [20] and $\theta(x)$ is the Heaviside step function

$$\theta(x) = \begin{cases} 1, & \mathrm{i}f \;\; x > 0, \\ \frac{1}{2}, & \mathrm{i}f \;\; x = 0, \\ 0, & \mathrm{i}f \;\; x < 0. \end{cases}$$

After they are guessed, it is quite straightforward to prove (39) and (40) by induction using the recurrence relation (38) and the following properties of the Euler polynomials:

$$\frac{\mathrm{d}}{\mathrm{d}x} E_n(x) = nE_{n-1}(x), \quad E_n(1-x) = (-1)^n E_n(x). \quad (41)$$

In particular, after rather lengthy but straightforward integration we get

$$\int_0^{1-u} K_{2n+1}(u_1, v)du_1 = K_{2n+2}(u,v) - X,$$

where

$$X = (-1)^{n+1} \frac{2^{2n}}{(2n+1)!} \left[E_{2n+1}\left(\frac{1+v}{2}\right) + E_{2n+1}\left(\frac{1-v}{2}\right) \right].$$

But

$$\frac{1-v}{2} = 1 - \frac{1+v}{2}$$

and the second identity of (41) then implies that $X = 0$.

Therefore the only relevant question is how (39) and (40) were guessed. Maybe the best way to explain the "method" used is to refer to problem 13 from the aforementioned book [1]. To demonstrate the cardinal difference between the ways problems are posed and solved by physicists and by mathematicians, Arnold provides the following problem for children:

"On a bookshelf there are two volumes of Pushkin's poetry. The thickness of the pages of each volume is 2 cm and that of each cover 2 mm. A worm bores through from the first page of the first volume to the last page of the second, along the normal direction to the pages. What distance did it cover?"

On a bookshelf there are two volumes of Pushkin's poetry. The thickness of the pages of each volume is 2 cm and that of each cover 2 mm. A worm bores through from the first page of the first volume to the last page of the second, along the normal direction to the pages. What distance did it cover?

Usually kids have no problems to find the unexpected correct answer, 4 mm, in contrast to adults. For example, the editors of the highly respectable physics journal initially corrected the text of the problem itself into: "from the last page of first volume to the first page of the second" to "match" the answer given by Arnold [1, 21]. The secret of kids lies in the experimental method used by them: they simple go to the shelf and see how the first page of the first volume and the last page of the second are situated with respect to each other.

The method that led to (39) and (40) was exactly of this kind: we simply calculated a number of explicit expressions for $K_n(u,v)$ using (38) and tried to locate regularities in these expressions.

Having (39) at our disposal, it is easy to calculate the integral in (37). Namely, because

We recover the celebrated formula (46):
$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1}}{(2n)!} \pi^{2n} B_{2n}$

$$K_{2n}(u,u) = (-1)^n \frac{2^{2n-2}}{(2n-1)!} \left[E_{2n-1}(u) + E_{2n-1}(0)\right], \tag{42}$$

and

$$E_{2n-1}(u) = \frac{1}{2n} \frac{\mathrm{d}}{\mathrm{d}u} E_{2n}(u), \tag{43}$$

we get

$$\mathrm{Vol}_{2n}(\delta_{2n}) = \int_0^1 K_{2n}(u,u)\,\mathrm{d}u = (-1)^n \frac{2^{2n-2}}{(2n-1)!} E_{2n-1}(0), \tag{44}$$

(note that $E_{2n}(0) = E_{2n}(1) = 0$.) But $E_{2n-1}(0)$ can be expressed in terms of the Bernoulli numbers

$$E_{2n-1}(0) = -\frac{2}{2n}(2^{2n}-1)B_{2n}, \tag{45}$$

and combining (31), (33), (44) and (45), we finally reproduce the celebrated formula

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1}}{(2n)!} \pi^{2n} B_{2n}. \tag{46}$$

5. The General Case of $\zeta(2n+1)$

The evaluation of $\zeta(3)$ can be also generalized straightforwardly. We have

$$\zeta(n) = \int_0^1 \frac{Li_{n-1}(x_1)}{x_1}\mathrm{d}x_1 = \int_0^1 \frac{\mathrm{d}x_1}{x_1} \int_0^{x_1} \frac{Li_{n-2}(x_2)}{x_2}\,\mathrm{d}x_2$$
$$= \iint_D \frac{\mathrm{Li}_{n-2}(x_2)}{x_1 x_2}\mathrm{d}x_1\mathrm{d}x_2.$$

Interchanging the order of integrations in the two-dimensional integral, we get

$$\zeta(n) = \int_0^1 \frac{\mathrm{Li}_{n-2}(x_2)}{x_2}\,\mathrm{d}x_2 \int_{x_2}^1 \frac{\mathrm{d}x_1}{x_1} = -\int_0^1 \frac{\ln(x_2)\mathrm{Li}_{n-2}(x_2)}{x_2}\,\mathrm{d}x_2. \tag{47}$$

Now we can repeatedly apply the recurrence relation (24), along with $\mathrm{Li}_1(x) = -\ln(1-x)$ at the last step, and transform (47) into

$$\zeta(n) = \int_0^1 \frac{\ln x_1}{x_1}\,\mathrm{d}x_1 \int_0^{x_1} \frac{\mathrm{d}x_2}{x_2} \cdots \int_0^{x_{n-4}} \frac{\mathrm{d}x_{n-3}}{x_{n-3}} \int_0^{x_{n-3}} \frac{\ln(1-x_{n-2})}{x_{n-2}}\,\mathrm{d}x_{n-2},$$

which after rescaling

$$x_2 = x_1y_2, \quad x_3 = x_2y_3 = x_1y_2y_3, \ldots, \quad x_{n-2} = x_{n-3}y_{n-2} = x_1y_2\cdots y_{n-2},$$

takes the form

$$\zeta(n) = \int_0^1 \frac{\ln x_1}{x_1}\,\mathrm{d}x_1 \int_0^1 \frac{\mathrm{d}y_2}{y_2} \cdots \int_0^1 \frac{\mathrm{d}y_{n-3}}{y_{n-3}} \int_0^1 \frac{\ln(1-x_1y_2\cdots y_{n-2})}{y_{n-2}}\,\mathrm{d}y_{n-2}. \tag{48}$$

Then the relation

$$\int_0^1 \frac{\mathrm{d}y_{n-1}}{1-x_1y_2\cdots y_{n-1}} = -\frac{\ln(1-x_1y_2\cdots y_{n-2})}{y_1y_2\cdots y_{n-2}}$$

shows that (48) is equivalent to the $(n-1)$-dimensional integral

$$\zeta(n) = -\int\cdots\int_{\Box_{n-1}} \frac{\ln x_1}{1-x_1\cdots x_{n-1}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}. \tag{49}$$

As in the previous case, (49) can be further transformed into

$$\zeta(n) = -\frac{2^n}{2^n-1}\int_{\square_{n-1}}\cdots\int \frac{\ln x_1}{1-x_1^2\cdots x_{n-1}^2}\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1},$$

or, in the more symmetrical way,

$$\zeta(n) = -\frac{2^n}{2^n-1}\frac{1}{n-1}\int_{\square_{n-1}}\cdots\int \frac{\ln(x_1\cdots x_{n-1})}{1-x_1^2\cdots x_{n-1}^2}\,\mathrm{d}x_1 \cdots\mathrm{d}x_{n-1}. \quad (50)$$

Let us now assume that n is odd and apply the Beukers–Kolk–Calabi change of variables (28) to the integral (50). We get

$$\zeta(2n+1) = -\frac{1}{2n}\frac{2^{2n+1}}{2^{2n+1}-1} \int_{\Delta_{2n}}\cdots\int \ln[\tan(u_1)\cdots\tan(u_{2n})]\mathrm{d}u_1\cdots\mathrm{d}u_{2n},$$

which is the same as

$$\zeta(2n+1) = -\frac{2^{2n+1}}{2^{2n+1}-1} \int_{\Delta_{2n}}\cdots\int \ln[\tan(u_1)]\,\mathrm{d}u_1\cdots\mathrm{d}u_{2n}.$$

By rescaling variables (defined in (34)), we can go from the polytope Δ_{2n} to the polytope δ_{2n} in this $2n$-dimensional integral and get

$$\zeta(2n+1) = -\frac{2^{2n+1}}{2^{2n+1}-1}\left(\frac{\pi}{2}\right)^{2n} \int_{\delta_{2n}}\cdots\int \ln\left[\tan\left(u_1\frac{\pi}{2}\right)\right]\mathrm{d}u_1\cdots\mathrm{d}u_{2n}. \quad (51)$$

Using the kernel $K_{2n}(u,v)$, we can reduce the evaluation of (51) to the evaluation of the following one-dimensional integral:

$$\zeta(2n+1) = -\frac{2\pi^{2n}}{2^{2n+1}-1}\int_0^1 \ln\left[\tan\left(\frac{\pi}{2}u\right)\right] K_{2n}(u,u)\,\mathrm{d}u. \tag{52}$$

But

$$\ln\left[\tan\left(\frac{\pi}{2}(1-u)\right)\right] = \ln\left[\cot\left(\frac{\pi}{2}u\right)\right] = -\ln\left[\tan\left(\frac{\pi}{2}u\right)\right],$$

which enables to rewrite (52) as

$$\zeta(2n+1) = -\frac{\pi^{2n}}{2^{2n+1}-1}\int_0^1 \ln\left[\tan\left(\frac{\pi}{2}u\right)\right]\left[K_{2n}(u,u) - K_{2n}(1-u,1-u)\right]\mathrm{d}u. \tag{53}$$

However, from (42) and (43) we have (recall that $E_{2n-1}(1-u) = -E_{2n-1}(u)$)

$$K_{2n}(u,u) - K_{2n}(1-u,1-u) = (-1)^n\,\frac{2^{2n-1}}{(2n)!}\,\frac{\mathrm{d}}{\mathrm{d}u}E_{2n}(u),$$

and the straightforward integration by parts in (53) yields finally the result

$$\zeta(2n+1) = \frac{(-1)^n\,\pi^{2n+1}}{4\left[1-2^{-(2n+1)}\right](2n)!}\int_0^1 \frac{E_{2n}(u)}{\sin(\pi u)}\,\mathrm{d}u. \tag{54}$$

This is exactly the integral representation for $\zeta(2n+1)$ found in [13]. Our earlier result (22) for $\zeta(3)$ is just a special case of this more general formula.

The beautiful result (54) generalizes the expression (22) for zeta(3).

6. Concluding Remarks: $\zeta(2)$ and Amoebas

It remains to clarify the origin of the highly non-trivial and miraculous Beukers–Kolk–Calabi change of variables (28). Maybe an interesting observation due to Passare [10] that $\zeta(2)$ is related to the amoeba of the polynomial $1 - z_1 - z_2$ gives a clue.

Amoeba of a Laurent polynomial P is the image of its zero locus under the log-mod map.

Amoebas are fascinating objects in complex geometry [22, 23]. They are defined as follows [24]. For a Laurent polynomial $P(z_1,\dots,z_n)$, let Z_P denote the zero locus of $P(z_1,\dots,z_n)$ in $(\mathbb{C}\backslash\{0\})^n$ defined by $P(z_1,\dots,z_n)=0$. The amoeba $A(P)$ of the Laurent polynomial $P(z_1,\dots,z_n)$ is the image of the complex hypersurface Z_P under the map

$$\mathrm{Log}:\ (\mathbb{C}\backslash\{0\})^n\to\mathbb{R}^n$$

defined through

$$(z_1,\dots,z_n)\to(\ln|z_1|,\dots,\ln|z_n|).$$

Let us find the amoeba of the following Laurent polynomial

$$P(z_1,z_2)=z_1-z_1^{-1}-i\left(z_2-z_2^{-1}\right). \qquad (55)$$

Taking

$$z_1=e^u\,e^{i\phi_u},\quad z_2=e^v\,e^{-i\phi_v},$$

we find that the zero locus of the polynomial (55) is determined by conditions

$$\cos\phi_u\,\sinh u=\sin\phi_v\,\cosh v,\ \ \sin\phi_u\,\cosh u=\cos\phi_v\,\sinh v.$$

If we rewrite these conditions as follows

$$x=\frac{\sinh v}{\cosh u}=\frac{\sin\phi_u}{\cos\phi_v},\quad y=\frac{\sinh u}{\cosh v}=\frac{\sin\phi_v}{\cos\phi_u}, \qquad (56)$$

we immediately recognize the Beukers–Kolk–Calabi substitution (8) and its hyperbolic version with the only difference that in (8) we had $0\le x,y\le 1$. However, from (56) we get

$$\cos^2\phi_u=\frac{1-x^2}{1-x^2y^2},\quad \cos^2\phi_v=\frac{1-y^2}{1-x^2y^2}, \qquad (57)$$

and

$$\cosh^2 u=\frac{1+y^2}{1-x^2y^2},\quad \cosh^2 v=\frac{1+x^2}{1-x^2y^2}. \qquad (58)$$

It is clear from (57) and (58) that we must have

$$x^2 \leq 1, \quad y^2 \leq 1.$$

Therefore, the amoeba $A(P)$ is given by relations

$$A(P) = \left\{ (u,v): \; -1 \leq \frac{\sinh u}{\cosh v} \leq 1, \;\; -1 \leq \frac{\sinh v}{\cosh u} \leq 1 \right\}, \tag{59}$$

and the hyperbolic version of the Beukers–Kolk–Calabi change of variables (8) transforms the unit square $\square$ into one-quarter of the amoeba (59). Then the analog of (9) indicates that $\zeta(2)$ equals one-third of the area of this amoeba.

The hyperbolic version of the Beukers–Kolk–Calabi change of variables (8) transforms the unit square into one-quarter of the amoeba (59). Then the analog of (9) indicates that ζ(2) equals one-third of the area of this amoeba.

As we see, the hyperbolic version of the Beukers–Kolk–Calabi change of variables seems more fundamental and arises quite naturally in the context of the amoeba (59). Trigonometric version of it then is just an area-preserving transition from the 'radial' coordinates (u, v) to the 'angular' ones (ϕ_u, ϕ_v).

Another amoeba related to $\zeta(2)$ was found in [10]. Although the corresponding amoeba $A(1 - z_1 - z_2)$ looks different from the amoeba (59), they do have the same area. The trigonometric change of variables used by Passare in [10] is also different from (8) but also leads to simple calculation of the area of $A(1-z_1-z_2)$ and hence $\zeta(2)$. Of course it will be very interesting to generalize this mysterious relations between $\zeta(n)$ and amoebas for $n > 2$ and finally disentangle the mystery. I'm afraid, however, that this game is already not for kids under fifteen.

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Suggested Reading

[1] V I Arnold, *Problems for Kids from 5 to 15*, MCNMO, Moscow, 2004, (in Russian).

[2] A M Yaglom and I M Yaglom, *Elementary Presentation of Non-elementary Problems*, problem 143, Gostekhizdat, Moscow, 1954, (in Russian).

[3] D Kalman, Six Ways to Sum a Series, *College Math. J.*, Vol.24, pp.402–421, 1993.

[4] K P Kokhas, Sum of inverse squares, *Mathematicheskoe Prosveshenie,* Vol.8, pp.142–163, 2004, (in Russian). See also informal notes by R Chapman, Evaluating ζ(2), http://www.secamlocal.ex.ac.uk/~rjc/etc/zeta2.pdf

[5] J Hofbauer, A simple proof of $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$ and related identities, *Am. Math. Mon*, Vol.109, pp.196–200, 2002.

[6] M D Hirschhorn, A simple proof that $\zeta(2)=\pi^2/6$, *Math. Intelligencer*, Vol.33N3, pp.81–82, 2011.

[7] N Lord, Yet another proof that $\sum\frac{1}{n^2}=\frac{1}{6}\pi^2$, *Math. Gazette*, Vol.86, pp.477–479, 2002.

[8] J T D'Avanzo and N A Krylov, ζ(n) via hyperbolic functions, *Involve*, Vol.3, pp.289–296, 2010.

[9] F M S Lima, New definite integrals and a two-term dilogarithm identity, *Indag. Math.*, Vol.23, pp.1–9, 2012.

[10] M Passare, How to compute $\sum\frac{1}{n^2}$ by solving triangles, *Am. Math. Mon.*, Vol.115, pp.745–752, 2008.

[11] F Beukers, J A C Kolk and E Calabi, Sums of generalized harmonic series and volumes, *Nieuw Archief voor Wiskunde*, fourth series, Vol.11, pp.217–224, 1993.

[12] N D Elkies, On the sums $\sum_{k=-\infty}^{\infty}(4k+1)^{-n}$, *Am. Math. Mon.*, Vol.110, pp.561–573, 2003; *Corr. ibid.*, Vol.111, p.456, 2004.

[13] D Cvijovic and J Klinowski, Integral representations of the Riemann zeta function for odd-integer arguments, *J Comput. Appl. Math.*, Vol.142, pp.435–439, 2002.

[14] M Aigner and G M Ziegler, *Proofs from THE BOOK*, Springer-Verlag, Berlin Heidelberg, third edition, p.38, 2004.

[15] D Cvijovic and H M Srivastava, Closed-form evaluations of certain definite integrals by employing the Cauchy integral theorem, *Numer. Algor.*, Vol.49, pp.129–141, 2008.

[16] D Cvijovic, Closed-form evaluation of some families of cotangent and cosecant integrals, *Integral Transform. Spec. Funct.*, Vol.19, pp.147–155, 2008.

[17] D Cvijovic, Closed-form evaluation of some families of definite tangent and secant integrals, *Integral Transform. Spec. Funct.*, Vol.18, pp.569–579, 2007.

[18] J Kubilius, Estimating the second central moment for strongly additive arithmetic functions, *Lithuanian Mathematical Journal,* Vol.23, pp.61–69, 1983.

[19] Z K Silagadze, Comment on the sums $S(n)=\sum_{k=-\infty}^{\infty}\frac{1}{(4k+1)^n}$, *Georgian Math. J.*, Vol.19, pp.587–595, 2012.

[20] See, B K Karande and N K Thakare, On the unification of Bernoulli and Euler polynomials, *Indian J. Pure Appl. Math*., Vol.6, pp.98-107, 1975; G Bretti and P E Ricci, Euler polynomials and the related quadrature rule, *Georgian Math. J.,* Vol.8, pp.447–453, 2001.

[21] V I Arnold, *Mathematics and Physics*, G Boniolo, P Budinich and M Trobok (Eds.) *The Role of Mathematics in Physical Sciences*, pp.225–233, Springer, Dordrecht, 2005.

[22] O Viro, WHAT IS an amoeba?, *Notices Amer. Math. Soc*., Vol.49, pp.916–917, 2002.

[23] M Passare and A Tsikh, Amoebas: their spines and their contours, *Contemp. Math*., Vol.377, pp.275–288, 2005.

[24] I M Gelfand, M M Kapranov and A V Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

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