Grothendieck and the Concept of Space

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A very brief introduction to Grothendieck's concepts of algebraic schemes and algebraic spaces.

1. A Promise and a Threat

Many mathematical texts begin with the assertion that what follows does not assume much by way of prior knowledge (this is the promise), but it does assume that the reader is 'mathematically mature' (which is the threat!). We do the same.

Before Grothendieck, many other mathematicians in the centuries gone-by have given their own versions of what 'geometry' is all about. We have all heard (and by now do not really believe) the grandmother's tale that 'geometry' is based on measuring ('-metry') the world ('geo-'). So, what is geometry actually about?

2. Not the Thing in Itself

From the beginning (as far back as Euclid), geometry has been about the relations between objects. Points that *lie on* lines, pairs of lines that *meet* (or do not meet) and so on. Grothendieck realised that the mathematical notion of 'category' can be used to capture this aspect of geometry.

A category is a systematic way of dealing with a certain class of mathematical constructions. The fundamental notions are those of objects of the category and the morphisms between objects of the category. Rather than give a definition, let us look at some examples.

1. The category of natural numbers: Its objects are natural numbers and a morphism from a to b rep-

resents the relation that a is less than or equal to b.

2. The category of linear algebra: Its objects are natural numbers and a morphism from a to b is a $b \times a$ matrix.

In both cases, we see that morphisms can be composed (in the first case by the transitivity of the relation $\leq$, and in the second case through matrix multiplication). This composition has useful algebraic properties like identity and associativity.

For the longest time, mathematicians have focused on the *internal structure* of the objects. The question is often posed as, "How is this geometrical object constructed?" Grothendieck pointed out that rather than analysing objects, we should concentrate on morphisms between them.

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We are less interested in the 'thing-in-itself' and more in the way each thing relates to other things!

3. Not Solving Equations

With the advent of Cartesian geometry, the focus of geometry was thought to have shifted to 'solving equations'. A Cartesian line is the locus of solutions (x, y) of an equation like $2x + 5y = 3$, an ellipse is the locus of solutions (x, y) of an equation like $2x^2 + 5y^2 = 3$, and so on.

However, solving equations is hard and we said that we would only do easy things! So, rather than consider the solution sets, let us consider a line as *defined* by $(x, y; 2x + 5y - 3)$, an ellipse as $(x, y; 2x^2 + 5y^2 - 3)$.

In other words, an object X of our category can be represented symbolically as $(x_1, \ldots, x_p; f_1, \ldots, f_q)$, where x_i denote the variables and f_j are the (polynomial) functions of these variables that vanish on our object.

If $Y = (y_1, \ldots, y_s; g_1, \ldots, g_t)$ is another object, a morphism from X to Y is a *substitution* of the form $y_l = h_l$, where h_l are functions of x_i so that the vanishing of g_k is *automatic* when the vanishing of the f_i is *given*[1].

[1] Algebraically, this amounts to the assertion that $g_k(h_1,...,h_s)$ is a linear combination of the form $\Sigma F_j f_j$, where F_j's are some polynomials in the variables x_i.

For example, we have a morphism from the 'point' $(x_1, x_2; x_1+1, x_2-1)$ to the 'line' $(y_1, y_2; 2y_1+5y_2-3)$; the substitution is $y_1 = x_1$ and $y_2 = x_2$. So solving equations is cleverly hidden within our notion of morphisms!

The category described above is the category of *Affine Schemes*[2].

[2] For those in the Mathematical High Galleries: Yes, the author is aware that it should actually be called the category of Affine Schemes of Finite Type – but let us gobble up all those extra four syllable phrases and cheat a little!

4. Quo(tient) Vadis

Riemann taught us that there is no one natural way to put Cartesian coordinates on our geometrical objects. (In physics, this is why we use 'frames of reference'). So, a geometrical object is often described by giving various coordinate-dependent descriptions and then providing equivalences between these descriptions.

Grothendieck realised that this meant that we need to describe geometrical objects that are *quotients* of affine schemes by equivalence relations.

For example, a triple (a_1, a_2, a_3) gives the parameters for a line $a_1x_1 + a_2x_2 = a_3$ in the plane[3]. Another triple (b_1, b_2, b_3) gives the *same* line if the 2×2 determinants of the matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$ vanish. So the *space* of lines in the plane is described as the *quotient* of the space of triples by the equivalence relation described by the affine scheme,

[3] We need to restrict our attention to triples for which either a_1 or a_2 is non-zero.

$$(a_1, a_2, a_3, b_1, b_2, b_3; b_1a_2 - a_1b_2, b_1a_3 - a_1b_3, b_2a_3 - a_2b_3)\,.$$

A geometrical object is often described by giving various coordinate-dependent descriptions and then providing equivalences between these descriptions.

We can ask whether the space of lines in the plane (the quotient by this equivalence) is *already* represented by an affine scheme; have we already answered this question? Sometimes, it is not so obvious whether or not we have.

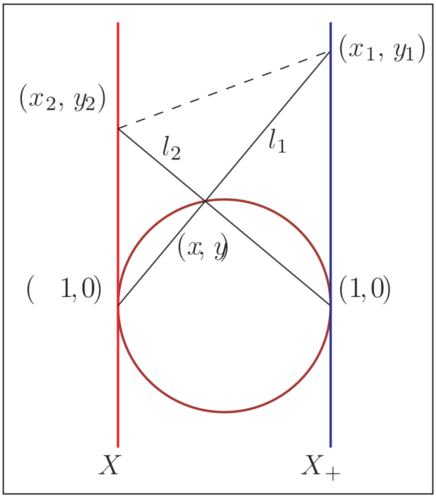

Figure 1.

For example, the affine scheme for the 'circle' $S^1 = (x, y; x^2+y^2-1)$ is the quotient of $X_+ = (x, y; x-1)$ and $X_- = (x, y; x+1)$ by the equivalence shown in *Figure* 1; the dashed line represents the equivalence between points on the blue line and the red line.

More explicitly, the equivalence between a point (x_1, y_1) in X_+ and a point (x_2, y_2) in X_- is given by the fact that the line l_1 joining (x_1, y_1) and $(-1, 0)$ and the line l_2 joining (x_2, y_2) and $(1, 0)$ meet at a point (x, y) on the circle[4].

[4] It is a simple, but instructive exercise in coordinate geometry to find the formula for this equivalence.

In this case, we *do* think of these two lines as *covering* the circle and so we *can* think of the circle as the quotient of two lines by the above equivalence relation.

5. Fiddler on the Cover

Through his work *Analysis Situs*, Poincaré pointed out that to realise the ideas of Riemann, one needs to introduce the notion of a covering (by open sets in Poincaré's context) to create new spaces as quotients by an equivalence relation (or 'patching').

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Grothendieck introduced the definition of (what we now call) a Grothendieck topology on a category by turning this idea on its head as follows.

Given a morphism $U \to X$, we obtain an equivalence relation R on U that represents the locus of pairs on U that have the same image in X[5]. A Grothendieck topology on the category is obtained by specifying those $U \to X$ where we wish to *declare* X to be the quotient of U by this equivalence relation.

[5] R is denoted as $U \times_X U$ and is called the fibre product of U with itself over X.

The covering notion has some natural properties: (a) The identity map $X \to X$ is a cover; (b) the cover of a cover is a cover. Once we have a Grothendieck topology, we can expand our category to include quotients that are not already present. The Grothendieck topology then extends to the expanded category; this makes quotient maps into coverings of the new spaces we have created.

For affine schemes, there are a few Grothendieck topologies that seem 'natural'. Zariski introduced (in an earlier avatar of affine schemes) a topology which is natural from an algebraic perspective. For example, the union of X_+ and X_- above is a cover of the circle S^1 in the Zariski topology. The resulting category of quotients is called the category of algebraic schemes[6].

[6] As before, the more complete name is the category of Algebraic (Pre-)Schemes of Finite type.

In some other contexts, one might wish to think of the 'doubling' morphism $S^1 \to S^1$ given by $(x, y) \mapsto (x^2 - y^2, 2xy)$ as a covering. Such ideas lead to the étale topology introduced by Grothendieck. The category of quotients was extensively studied by Artin and Knudsen and is called the category of algebraic spaces.

6. Ten Thousand Pages Later ...

The task of completing the above ideas into a mathematically complete theory took Grothendieck (and his students and collaborators) a number of years. In fact, each of the sections above requires a book by itself!

"Has G proved any theorems?" Indeed, in the process of formulating these ideas, a number of results in algebra and differential calculus were formulated and proved. Further notions need to be introduced; calculus can be studied via a notion of infinitesimal equivalence relation; ideas from algebraic topology need to be defined and extended, and so on. As a result of the clarity obtained from this viewpoint, a number of classical questions about geometry have been answered. The books *EGA* (*Éléments de Géométrie Algébrique)* and *SGA* *(Séminaire de Géométrie Algébrique*) spell out many of the details.

The idea of the category of quotients leads to the important notion of 'Topos' which has applications in such diverse areas as logic and programming languages.

Grothendieck himself further extended the idea of a quotient and this led to n-stacks as a further development of the concept of a space. A different approach led Quillen to his notion of a simplicial model category. Simplicial spaces, n-stacks and the relations between them are active areas of research.

Acknowledgement

I was dragged kicking and screaming into Grothendieck's world of categories by Nitin Nitsure when both of us were graduate students at the School of Mathematics, TIFR, Mumbai. He would agree with me that the ideas of Grothendieck continue to fascinate and intrigue us.

Suggested Reading

[1] **The Red Book of Varieties and Schemes, David Mumford, Springer, 2nd edition, 1999.**

[2] **Basic Algebraic Geometry, Igor Shafarevic, Springer, 2nd edition, 1994.**

To see other uses of these ideas:

[3] **Sheaves, Geometry and Logic: A First Introduction to Topos Theory, Mac Lane and Moerdijk, Springer, 2nd edition, 1994.**

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