A Glossary of Some Mathematical Terms

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M S Narasimhan's article in this issue on page 482 contains several mathematical terms. In this glossary, we try to briefly explain some of them.

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(1) 'Algebraic geometry' is the study of solutions of systems of polynomial equations in a number of variables. Of course, the study of roots of a polynomial in a single variable falls within this scope.

Further, this expands on the solutions of systems of linear equations and the study of 'conics' (solutions of quadratic equations in two variables).

(2) 'Number theory' means different things to different people. It is primarily the study of (counting) numbers and their properties. In the course of this study, one is led to the study of other number systems and their properties as well.

(3) 'Topology' is often described as the study of 'shape' or 'rubber-sheet' geometry. However, within mathematics, there are two subjects – one addressing 'point-set' topology, which is primarily a study of certain systems of subsets of a set (called a topology on that set), and the other dealing with 'algebraic' topology which is the study of 'shape' through algebraic invariants that characterize the presence of non-trivial shape-like properties.

(4) 'Homological algebra' grew out of the study of algebraic invariants associated with shape and the study of certain properties of groups. In the seminal work on this topic by Cartan and Eilenberg, it was shown that what was needed was a vast systematic generalization of the rank-nullity[1] theorem of linear algebra.

[1] The basic rank-nullity theorem says that for a linear map from a vector space V to a vector space W, the sum of its rank and its nullity gives the dimension of V.

(5) 'Functional analysis' is the study of functionals (functions on

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spaces of functions). Elementary examples include the integral of a function or the value of a function. With the advent of point-set topology, it became possible to study these infinite dimensional spaces systematically. Such spaces play an important role in quantum mechanics and more generally in the solution of differential equations.

(6) 'Weil conjectures' (which are now theorems due to Dwork, Grothendieck, Deligne and others) are still known by their original name. They provide a link between the divisibility of values of polynomial functions by primes and the topology of the zeroes of these functions over the field of complex numbers.

(7) 'The Mordell conjecture' (which is now a theorem due to Faltings) is still known by its original name. The theorem shows that the number of rational solutions to an equation in two variables is finite if the solution set has 'at least two holes' in a certain precise sense as described in terms of algebraic topology (see (3)).

(8) 'Modularity conjecture' (which is now a theorem due to Wiles–Taylor) asserts that any equation of the form $y^2 = x^3 + ax + b$ with a and b integers (or rational numbers) can be given parametric solutions in terms of certain special functions called modular functions on the upper half-plane. This was conjectured by Shimura and Taniyama and also appeared in a problem posed by Weil. Frey proposed the study of the curve $y^2 = x(x - a^n)(x + b^n)$ in connection with $a^n + b^n + c^n = 0$, the famous Fermat equation. Serre, Ribet and others were able to develop Frey's ideas to show that the modularity of such a curve would lead to a contradiction, thus giving a proof of Fermat's last theorem.

(9) 'Measure theory' develops a consistent way to assign a measure to some subsets of a certain fixed space. This theory lies at the intersection between the theory of integration and probability theory. Probability can be formulated via a theory of measure where the measures lie between 0 and 1.

(10) 'Lebesgue integration' is the best known way of extending

the classical theory of integration to large classes of functions in a consistent way. It was developed by Lebesgue in the late 19th century and is a fundamental idea in modern analysis.

(11) While most people would feel that they know what 'geometry' is about, they would only be partially right. Just as someone without a trained eye would not be able to see 'painting' in the works of the modern (and later) schools of art, most people would not be able to recognize any 'geometry' in the kind of mathematics being written by modern geometers. However, they can be reassured that (just as with art) there is a clear historical lineage and the source of inspiration is indeed similar!

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(12) 'Analysis' means a lot of things in different contexts. In mathematics, the roots of analysis lie in the fundamental ideas of Weierstrass, Dedekind, Cauchy, Cantor and others. They understood that one must analyze the classical study of functions and numbers in terms of the fundamental notions of convergence formulated in terms of the infamous ε and δ, and derive all required results as a consequence. This is what 'modern' analysis deals with.

(13) 'Sheaves' are a concept invented by Leray as a means of understanding functions that take values in spaces that change with the point of evaluation. A sheaf F over space X can be defined as a map $F \to X$ that induces (at each point f of F) a homeomorphism (topological isomorphism) between a neighborhood of f and a neighborhood of its image in X. (This is *not* the most useful definition of a sheaf!)

(14) There is nothing very spiritual or rainbow-like about 'spectral sequences'! In many cases, an algebraic invariant $w(X)$ associated (by algebraic topologists) with a space X can be understood in terms of maps from X to a certain 'virtual' space $E(w)$. These virtual spaces were referred to as 'spectra'. The study of such spaces using the tools of homological algebra led to the formalism of spectral sequences, which, like most homological algebra, allow one to make iterated complicated calculations

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without worrying about the geometric meaning of that which is being calculated.

(15) During his study of 'functional analysis', Schwarz found that it is convenient to treat as 'generalized functions', certain functionals that are not associated with functions but behave like functions in many ways; he called them 'distributions'. The most famous one is the 'Dirac delta' which takes a function to its value at a point.

(16) 'Topological vector spaces' are the principal object of study in 'functional analysis' as explained above. Such spaces are typically obtained by putting a suitable topology on a (linear) space of functions. More such spaces are obtained by 'topologising' the standard constructions in linear algebra such as duals, linear transformations and so on.

(17) A collection of subsets (called open subsets) of a certain space is called a (classical) 'topology' on the space if this collection satisfies some natural axioms. The study of the resulting structure is what is called point-set topology.

(18) A 'nuclear space' is a special kind of topological vector space that shares many of the properties of finite dimensional vector spaces. In particular, even when such a space is infinite dimensional, some important notions such as traces and spaces of bilinear operators can be defined.

(19) 'Scheme' is a concept introduced by Grothendieck to supersede the notion of an algebraic variety as the fundamental object of study in algebraic geometry. We can roughly imagine it to be a space which has a number of components glued together, where each component is an (infinitesimal) thickening of an algebraic variety. (Note that the formal definition allows for somewhat more generality.) An algebraic scheme can be also be thought of as the 'locus' of solutions of a system of polynomial equations in a number of variables. (Here, the term 'locus' needs to be understood in the sense of solutions in various number systems or 'rings').

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(20) The concept of a 'ring' generalizes the classical concept of a number system; it is a collection of entities where addition, subtraction and multiplication of these entities make sense. For example, one can take the entities to be all the square matrices of a fixed size. Another example is the collection of all polynomials made using some set of variables. Multiplication in the former case is (for size at least two!) not commutative, while in the latter case, multiplication *is* commutative. We have a 'commutative ring' when multiplication is commutative.

(21) In a number of cases, a space X is not constructed directly. Rather it is the locus of equivalence classes in some other space U. In this case, 'descent theory' is a mechanism to study properties of the space X in terms of the properties of the space U.

(22) One of the achievements of algebraic geometry over the last hundred or so years has been the study of 'moduli spaces'. These are spaces that *parametrize* certain types of geometric objects. (For example, the Mobius strip can be conceptualized as the space of all lines in the plane). An important step in this study is to construct a space of all geometric objects of a particular type *inside* a fixed space. The notions of 'Hilbert and Quot schemes' were introduced by Grothendieck as a way of realizing such constructions in general.

(23) In many contexts, one can find a way to iteratively construct approximations of arbitrarily high order to the solution that one wishes to find. Grothendieck found a general way to recognize such contexts (in algebraic geometry) through his 'formal existence theorem'.

(24) The work of Picard in the late 19th century provided the construction of a certain group (which was also a variety) called the Picard variety associated with an algebraic surface. His construction used analysis rather than algebra and thus limited its utility, for example in applications to problems in number theory. One of the demonstrations of the power of the techniques of schemes, descent theory and Hilbert schemes was the completely

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algebraic construction (by Grothendieck) of 'Picard schemes' associated with a large class of schemes.

(25) A 'covering' of a space X is a space Y with a map to X so that every point of X has the same number of points in Y that lie over it; this collection of points in Y is called the fiber over the point of X. An easy example is provided by the map from the circle to itself that folds the circle over itself a finite number of times; mathematically, if we think of the circle as points $(\cos(t), \sin(t))$, the map is given by sending this to $(\cos(nt), \sin(nt))$.

(26) For a covering map $Y \rightarrow X$, the maps $Y \rightarrow Y$ which take fibers to fibers are called covering transformations. These form a group. There is a covering (called the *universal* covering) which lies over all other coverings. The group of covering transformations of the universal covering is called the 'fundamental group' of the space X. It is one of the first algebraic invariants introduced in algebraic topology.

(27) Grothendieck (and Giraud) introduced the notion of 'topos' as a way of talking about topology *without* the 'crutch' of set theory. The formulation of topos is in the language of categories and functors; hence one can work with topological concepts in a large number of contexts where there is no natural topology on the 'underlying set'. For example, we may be looking at solutions of equations over finite fields. In this case, the underlying sets are finite and hence do not apparently have an interesting topology.

(28) Singular cohomology (formulated by Eilenberg and Steenrod) is one of the important algebraic invariants attached to a topological space. By extending the key ideas needed for this theory to a topos, Grothendieck was able to define a cohomology theory (which he called 'étale cohomology') which assigned algebraic invariants to any algebraic scheme. These invariants were essentially the same as singular cohomology in the case when the scheme had a 'classical' topological space associated with it. At the same time, the invariants were defined for varieties of finite fields and more general rings, and had essentially the same behavior as they did in the classical case.

(29) The formula of Riemann–Roch calculates the (alternating sum of the) dimensions of some naturally occurring vector spaces of functions on a Riemann surface in terms of some numerical invariants of the Riemann surface (which can be seen as topological invariants as well).

(30) 'λ-adic integers' are obtained from integers by completing them using a notion of a distance that depends on divisibility by powers of a prime l. 'λ-adic numbers' are ratios of l-adic integers.

(31) 'λ-adic cohomology' is an important special example of étale cohomology where we look at the limits of algebraic invariants which are annihilated by a fixed power of a prime l. This cohomology is then studied as a module over l-adic integers.

(32) For a sequence of numbers a_n, one can study the 'Dirichlet series', $L(a,s) = \sum_n \frac{a_n}{n^s}$, which allows one to study some common arithmetic properties of a_n along with their asymptotic behavior.

(33) 'L-functions' are certain special types of Dirichlet series that arise in the theory of group representations as well as in the study of algebraic schemes defined by equations with integer coefficients. The inter-relation between these two types of L-functions is the aim of 'the Langlands programme'.

(34) When a (differential) equation has singular points, and solutions are 'followed' in a closed path that avoids the singularity, we obtain a linear transformation on the space of solutions. This phenomenon goes under the name 'monodromy'.

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(35) 'Grothendieck–Riemann–Roch theorem' is a vast generalization of the Riemann–Roch formula to higher dimensions and to the 'relativized version' case. The formulation of the statement required the invention of K-theory and its relativisation introduced a new method of proof, all of which established Grothendieck as the mathematician whose lead was worth following!

(36) 'Algebraic varieties' are reduced irreducible algebraic schemes. In other words, they have only one component and this component has no thickening. Before Grothendieck introduced schemes, these were considered to be the primary object of study in algebraic geometry and in some sense they still are. However, in the course of studying them, we have no choice (as Grothendieck showed) except to use the broader notion of schemes.

(37) When dealing with certain types of mathematical objects, we restrict the kinds of transformations between them that are 'allowed'. Elements of this restricted class are called morphisms. In the case of algebraic schemes (or varieties), morphisms are limited to those which are given by polynomial functions.

(38) In the late 19th century and the early 20th century, mathematical notions were formulated in the language of set theory and an attempt was made to put set theory at the foundation of mathematics. All interesting mathematical concepts were then formulated as 'sets with structure'. This is the way in which most mathematical textbooks continue to introduce notions like groups, rings, manifolds, topological spaces, and so on.

(39) A 'category' encapsulates the notion of a certain 'type' of mathematical object (such as group) and morphisms between such types of objects. The study of such objects then becomes a study 'within the category'. In terms of sets with structure approach, we can say that the category has objects which are those types of structures and morphisms are limited to those which carry forward the structure. For example, we have the category of groups with morphisms being group homomorphisms, topological spaces and morphisms being continuous maps, and so on.

'Dévissage' is seen as a way of building up objects from simpler objects. In reverse, it can be seen as taking apart a complicated object to examine its construction in terms of simpler ones.

(40) It is often necessary to relate objects (and morphisms) of one category to objects (and morphisms) of another. The seminal example from algebraic topology is the one that associates the fundamental group of a topological space to that topological space. Such associations are called 'functors' between the categories if they satisfy some natural conditions.

(41) 'Dévissage' is seen as a way of building up objects from simpler objects. In reverse, it can be seen as taking apart a complicated object to examine its construction in terms of simpler ones.

(42) Grothendieck showed how one can associate an abelian group (called its K-group) to a certain type of category. This construction is a vast generalization of the construction of integers from the notion of whole numbers or the notion of fractions from the (multiplicative) notion of natural numbers. It is the cornerstone of Grothendieck's generalization of the Riemann–Roch formula.

(43) Grothendieck's construction of the K-group of a category was later seen as the first in a sequence of such algebraic invariants associated with categories. The full generalization of Grothendieck's construction is due to Quillen and the corresponding K-groups are known as Quillen's higher K-groups

(44) The fundamental problem of singularities is that certain functions appear to misbehave at some points. In the context of algebraic geometry, such behavior can be 'resolved' through (repeated application of) a procedure called the 'blowing-up' of the underlying space. This is a celebrated theorem proved by Hironaka in the 1960s.

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(45) The theory of motives is the pot of gold at the end of the rainbow that is the study of algebraic geometry! It attempts to link the topology of the solutions of a system of equations in a precise way with the number of solutions over finite fields as well as the number-theoretic properties of the algebraic invariants of the locus of solutions.

(46) The 'standard conjectures' are still unproven and are part of Grothendieck's programme for the theory of motives. Proving the conjectures would give an elegant proof of Weil conjectures which avoids 'tricks'.

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