From Navigation to Star Hopping: Forgotten Formulae

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One of the great challenges faced by navigators of yesteryears was the calculation of distances when sailing in the sea. They made extensive use of trigonometry to help them in this endeavour. Trigonometry provided them with quicker, more accurate methods, and formulae, for determining distances. The same formulae found application in a totally different field – astronomy, for instance in finding the angular separation between stars or between planets and stars. What is amazing is the striking similarity between solutions developed by people from different locations, backgrounds and even times.

Many formulae and trigonometric identities came into being originally to facilitate navigators. However, when they are taught in class, the practical use to which they had been put is not discussed. Some of the formulae have been forgotten simply because they had been devised for a specific purpose which is not relevant anymore. One such forgotten identity is the following which expresses the minimum distance between two points as a function of their latitude and longitude :

$$\text{hav} s = \text{hav}\Delta\phi + \cos\phi_1 \cos\phi_2 \text{hav}\Delta\lambda\,, \qquad (1)$$

where s is the angular separation (which multiplied by radius of earth is the minimum distance on the sea/land) between two stations with longitudes and latitudes as λ_1,ϕ_1 and λ_2, $\phi_2, \Delta\phi$ is the difference in latitudes and $\Delta\lambda$ is the difference in longitudes.

Keywords

Trigonometry, haversine, navigation, Indian astronomy, *Jyotpatti* of Bhaskaracharya II.

Function 'hav' is an archaic trigonometric ratio. It stands for 'half versed sine' or 'haversine' and abbreviated to

Box 1. Sir James Inman

Sir James Inman (1776–1859) was a mathematician and navigated extensively; when he travelled to the Middle East and Syria he did not miss the opportunity to learn Arabic. Later he joined the Royal Naval College at Portsmouth as a Professor of Nautical Mathematics and wrote a book *Navigation and Nautical Astronomy for Sea-men* in 1821 with tables known as Inman's nautical tables. This was extensively used by all navigators.

'hav'. It is defined as follows:

$$\operatorname{hav}(\theta) = \frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}\,. \tag{2}$$

This was introduced by James Inman (*Box* 1) in the 18th century. Let us first understand it in the context of plane triangles.

An interesting application of this relates to the solution of an oblique triangle. Consider the familiar rule which relates the lengths of the sides of a triangle to the cosine of one of its angles:

$$a^2 = b^2 + c^2 - 2bc\cos A\,. \tag{3}$$

This can be rewritten as

$$-\cos A = a^2 - (b^2 + c^2)/2bc \tag{4}$$

or

$$\frac{1-\cos A}{2} = \frac{a^2-(b-c)^2}{4abc}\,. \tag{5}$$

We can express (3) in terms of haversines as

$$a^2 - (b-c)^2 = 4bc\,\operatorname{hav}A \tag{6}$$

or, rearranging,

$$\operatorname{hav}A = \frac{(s-b)(s-c)}{bc}\,, \tag{7}$$

where s is half the perimeter of the triangle, i.e., $s = (a+b+c)/2$.

In the case of a right-angled triangle with c as the hypotenuse ($c^2 = a^2 + b^2$), it can be shown that

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{c-b}{2c}}\ . \qquad (8)$$

Squaring both sides

$$\sin^2\left(\frac{A}{2}\right) = \frac{c-b}{2c} = \mathrm{hav}(A)\ . \qquad (9)$$

Using the square of the sine function avoids the determination of square root, which must have been a tedious procedure without a calculator in old times. The formula had many takers and was extensively used.

In the third edition of the book, *Navigation and Nautical Astronomy for Sea-men*, in 1835, Sir James Inman introduced half-verse-sine as haversine, basically for calculating distances on the sea, simplifying the formulae usingspherical trigonometry. The formula is written down as

$$\mathrm{haversine}\left(\frac{d}{r}\right) = \mathrm{haversine}(\phi_2 - \phi_1) + \cos(\phi_1)\cos(\phi_2) \times \mathrm{haversine}(\lambda_2 - \lambda_1)\ . \qquad (10)$$

In this formula, d is the distance between two points (along a great circle of the sphere); r is the radius of the sphere (earth); ϕ_1, ϕ_2 are latitudes of points 1 and 2 and, λ_1, λ_2 are longitudes of points 1 and 2.

He further derived a law of haversines as

$$\mathrm{haversine}\,(c) = \mathrm{haversine}\,(a-b) + \sin a \sin b\, \mathrm{haversine}\,(c) \qquad (11)$$

from the law of cosines.

This formula, the law and the name of James Inman vanished in the subsequent trigonometry textbooks for reasons not known.

Box 2. Roger Sinnott

Roger W Sinnott, an astronomy graduate from Harvard, served on the editorial board of the monthly magazine *Sky and Telescope*. With his hands-on experience on telescopes from his school days, he managed the telescope section of the magazine with exhaustive discussions on telescopes, accessories and novel optical designs. In 1984, he introduced the section on computers for astronomical calculations for the future events as well as the past historical records. Roger co-authored the two-volume *Sky Catalogue 2000.0* and *NGC 2000.0*, which list 13,226 nebulae and star clusters and many more deep-sky objects accessible with an amateur telescope. Another work *Millennium Star Atlas*, co-authored by him has stellar brightness and proper motions from ESA's Hipparcos Astrometry Mission apart from stellar positions. The minor planet numbered 3706 is named after him recognising his contributions to the field of observational astronomy.

Star Hopping

A small note [2] by R W Sinnott appeared in the August 1984 issue of the magazine *Sky and Telescope* recalling the forgotten trigonometric function as a very good tool for navigating amidst the stars in the sky (*Box* 2). He even recommends that it should be re-introduced in the textbooks of today.

The standard formula for finding $\boldsymbol{s}$, the angle between two stars with right ascension[1] α_1, α_2 and declination δ_1, δ_2 is the following:

$$\cos s = \sin \alpha_1 \sin \alpha_2 + \cos \delta_1 \cos \delta_2 \cos \Delta\alpha \ , \qquad (12)$$

where, $\Delta\alpha = \alpha_1 - \alpha_2$. The difficulty in using this has been discussed by Sinnott in terms of the limitations of the computing tool, especially while dealing with small angles such as the fraction of an arc minute. In such cases, the spherical triangle is approximated to a plane triangle and then the theorem of Pythagoras comes to rescue. However, the use of the haversine formula instead of equation (12) will fetch better results.

$$\operatorname{hav} s = \operatorname{hav}\Delta\delta + \cos \delta_1 \cos \delta_2 \operatorname{hav}\Delta\alpha \ . \qquad (13)$$

[1] Right ascension and declination are the coordinates for celestial bodies; Right ascension is measured eastwards and usually represented in hours, minutes and seconds. Declination is the north-south coordinate expressed in degrees, arcminutes and arcseconds. The reference point is the First Point of Aries, the point of intersection of celestial equator and the ecliptic.

Since this is an unfamiliar identity, one has to work with the $\sin^2\left(\frac{A}{2}\right)$ tool on calculators and computers. Thisway, one can compute the angular separation to fractions of an arc second. Sinnott provided a sample program for getting the angular separation between Mizar (α3:21:54.953, δ55:11:09.24) and Alcor (α 3:23: 13.544, δ55:14:52.78). Using (12) the separation is calculated as 722.76" using a simple calculator with 7 digit accuracy. Using (13), the separation is 708.69", which is much closer to the correct value.

Thus, it turns out that operations with haversines are simpler and yield better results.

From the Pages of History

Although the concept of haversine and versed sine were coined by James Inman in 1835, we find that they were already in use in India. Almost all texts of Indian astronomy use these ratios. Here is an example from 12th century AD. *'Jyotpatti'* is an appendix in the classic text *Siddhanta Shiromani*[2] by Bhaskaracharya II (*Box* 3) at the end of the chapter called *'Goladhyaya'*. It provides a formula for haversine.

Let us start with *Jya*, which is the equivalent of sine and is defined for a circle of radius R as $Jya = R\text{sine}\theta$.

[2] *Siddhanta Shiromani*, authored by Bhaskaracharya II, whose 900th birth anniversary year was celebrated in 2014, is well cited by all astronomers of later centuries. It covers all the mathematical details and his own commentary on it is called *Vaasana Bhaashya*. The derivation of trigonometric identities are described as an appendix called *Jyotpatti*. *Jya* is the sine ratio; *utpatti* is derivation. The chapter describes the methods to estimate sine ratios for almost all values of angles.

Box 3. Bhaskaracharya II

Bhaskaracharya II (born 1114 CE) is well known for his contributions to astronomy and mathematics. His astronomical treatise *Siddhanta Shiromani* and mathematical work *Lilavati* are very popular. *'Lilavati'* has more than 35 commentaries. Bhaskara's text on algebra is the *'Bija-ganita'* which has discussions on the indeterminate equations of the first degree *'Kuttaka'* and of the second degree *'Varga-prakriti'*. (Many historians of mathematics object to this being called 'Pell's equation'!) Hankel, the famous German mathematician praises Bhaskara's *'Chakravaala'* method as the finest thing achieved in the theory of numbers before Lagrange. *Jyotpatti* deals with trigonometry defining *Jya* (sine), *Kotijya* (cosine) and *Utkramajya* (versedsine). Bhaskara obtained these trigonometric ratios for the standards angles of 30, 45, 60, 36 and 28, (all in degrees) by inscribing regular polygons in a circle of radius R. He further developed these results for multiple angles using formulae for addition and subtraction.

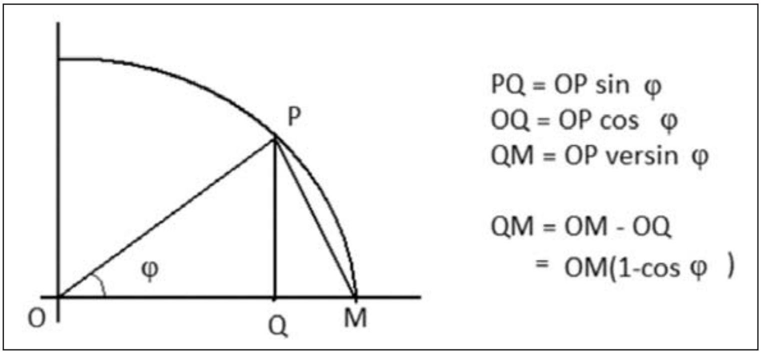


Figure 1. Definitions of trigonometric ratios. The trigonometric ratio *utkramajya* is translated as versed sine or verse sine or versin or simply verse by different translators of Sanskrit texts in the 18th and 19th centuries.

Apart from cosine, which is called *Kotijya* (cosine), another ratio called *Utkramajya* also is defined. This is translated into English as versed *R*sine. The definitions are as shown in *Figure* 1.

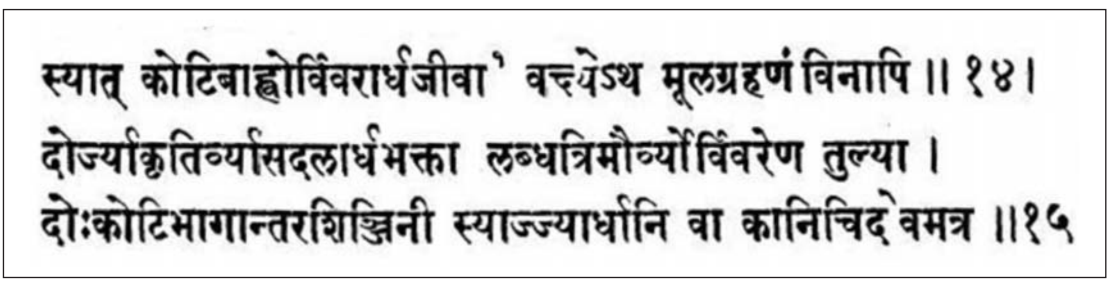
It is easily shown (see *Figure* 1) that

$$Utkramajya = \text{versed}\, R\text{sine}\phi = 1 - \cos\phi. \qquad (14)$$

In *Jyotpatti (jya+utpati*, meaning production of sine tables) Bhaskaracharya describes various methods for getting the values of the sine. As one of the methods to avoid square roots, he derives an expression for $\sin^2\left(\frac{A}{2}\right)$ in terms of *Utkramajya.*

The derivation follows from the text shown in *Figure* 2: Verse number 14 ends with the statement, "Now I will describe a method to avoid determination of square roots". All the previous methods invariably had the 'disadvantage' of determination of square root – a logic reflected in the book by James Inman as well.

Figure 2. The verses numbered 14 and 15 in *Jyotpatti*.


स्यात् कोटिबाह्वोर्विवरार्धजीवा' वक्ष्येऽथ मूलग्रहणं विनापि ॥ १४ ।
दोर्ज्याकृतिर्व्यासदलार्धभक्ता लब्धत्रिमौर्व्योर्विवरेण तुल्या ।
दोःकोटिभागान्तरशिञ्जिनी स्याज्ज्यार्धानि वा कानिचिदेवमत्र ॥१५

The verse when translated reads as, "The square of the *jya* of an arc is divided by half the radius; the difference between the quotient and the radius is equal to the *jya* of the difference between that arc and its complement."

Let us write this as an equation:

Square of *jya* $\{= R\text{sine}^2\phi\}$

divided by half the radius $\{ = R\text{sine}^2\phi \ /R/2\}$

minus the radius

is equal to *jya* of the difference between the arc and its complement

$\{= R\text{sine}(\phi - (90 - \phi)) = -R\text{sine}(90 - \phi - \phi)\}$

i.e.,

$$-R\text{sine}(90 - \phi - \phi) = \{R\text{sine}^2\phi/(R/2)\} - R \qquad (15)$$

which is the same as $\cos 2\phi = 1 - \sin^2\phi$.

For a proof of this, consider the trigonometric ratios for the angle ϕ = POX (*Figure* 3). Draw POQ also equal to ϕ. D is the midpoint of QX. Therefore,

$$QX = 2XD$$
$$QX^2 = 4XD^2$$
$$\text{Also,} \quad QX^2 = QT^2 + TX^2 \qquad (16)$$

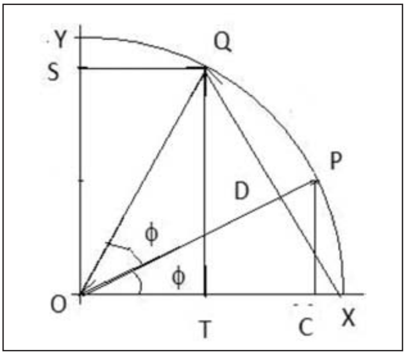


Figure 3. Construction for the derivation of $\sin^2\left(\frac{\varphi}{2}\right)$.

$$
\begin{aligned}
&= QT^2 + (OX - OT)^2 \\
&= QT^2 + OX^2 + OT^2 - 2OXOT\ . \qquad (17)
\end{aligned}
$$

Or, using (16)

$$
\begin{aligned}
OT &= \{OX^2 + (QT^2 + OT^2) - QX^2\}/2OX \\
&= \{R^2 + R^2 - QX^2\}/2R \\
&= \{2R^2 - QX^2\}/2R \\
&= \{2R^2 - 4XD^2\}/2R \\
&= R - \{2XD^2/R\}\ . \qquad (18)
\end{aligned}
$$

Since, $OT = QS = R\sin(90 - 2\phi)$ and $XD = R\sin\phi$, (18) can be rewritten as

$$R\sin(90 - 2\phi) = R - \{R^2 \sin^2\phi/(R/2)\}.$$

This is the same as equation (15) described in the verse number 15.

It turns out that the same definition was suggested by James Inman (2).

Thus it is interesting to note that for an easier estimate of the angular separations, exactly similar proceedures have been utilised at different epochs and at different geographical locations.

Suggested Reading

[1] G W Evans, *Archives of American Mathematical Society*, Vol.26, http://archive .org/details/jstor-2973146

[2] R W Sinnott, Virtues of the haversine', *Sky and Telescope*, Vol.68, No.2, p.159, August, 1984

[3] Venugopala Herur, *Jyotpatti*, Rajasthan Sanskrit Prathishtan, 2006.

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