Trigonometric Characterization of Some Plane Curves

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There is a way to describe a family of plane curves different from that using Cartesian or polar co-ordinates. This is a trigonometric equation involving two angles. In this article, we highlight the fact that trigonometric equations are convenient to describe certain one-parameter families of plane curves. In some cases, the trigonometric form makes it easier to characterize the family than if one were to use the equivalent Cartesian or polar representations.

1. Introduction

The study of one-parameter families of plane curves is a well-known, interesting and important topic in plane analytic geometry. These curves can be represented by an equation of the type $f(x, y, a) = 0$ or $g(r, \theta, b) = 0$, where a, b are real parameters. Here, (x, y) and (r, θ) denote respectively the Cartesian and the plane polar coordinates [1, 2].

There is another way of representing a family of plane curves. It involves two angles related to a point on a curve: θ, the vectorial angle of the point with reference to a pole and a polar axis, and ψ, the angle made by the tangent to the curve at that point with the polar axis (*Figure* 1). A relation between $\tan\theta$ and $\tan\psi$ gives the *trigonometric equation* of the family of curves.

In this article, trigonometric equations of some known plane curves are deduced and it is shown that these equations reveal some geometric characteristics of the families of the curves under consideration. In Section 2, families of conics are discussed, and in Section 3, some

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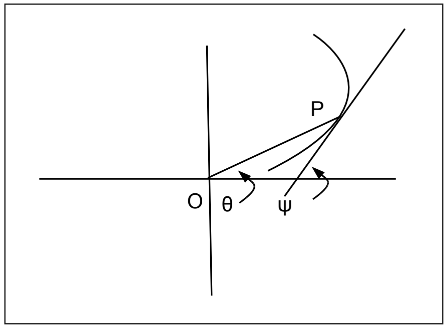
Plane curves, trigonometric equations.

Figure 1.



other plane curves. In Section 4, a few trigonometric equations are considered which lead to known families of plane curves.

2. Conics

Consider the equation

$$x^2 + \lambda y^2 = 1 \qquad (\lambda \in \mathbb{R} : \text{parameter}). \tag{2.1}$$

It is well known that this represents an ellipse if $\lambda > 0$ (the case $\lambda = 1$ represents a circle, which is a special case of an ellipse) and a hyperbola if $\lambda < 0$. For $\lambda = 0$, it represents a pair of parallel lines, $x = \pm 1$, which is a degenerate conic.

Let $\lambda \neq 0$. To deduce the trigonometric equation, we differentiate (2.1) with respect to x, to get

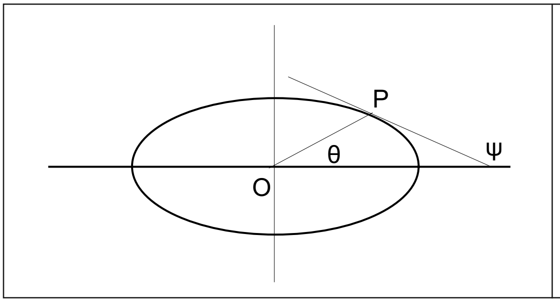
$$yy'/x = -\frac{1}{\lambda},$$

where $y' = \mathrm{d}y/\mathrm{d}x$. This relation can be written as

The product of the tangents of the two angles is a negative constant for an ellipse (– 1 for a circle), and a positive constant for a hyperbola.

$$\tan\theta \tan\psi = -\frac{1}{\lambda}. \tag{2.2}$$

Thus, the product of the tangents of the two angles is a negative constant for an ellipse (*Figure* 2) (-1 for a circle), and a positive constant for a hyperbola (*Figure* 3).



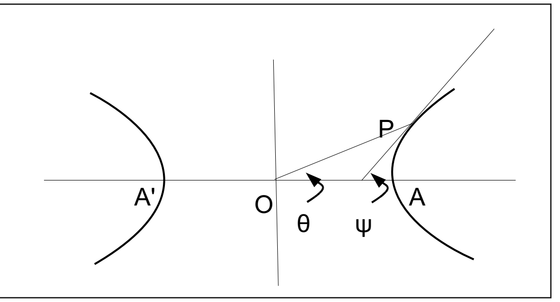


Figure 2. (left)
Figure 3. (right)

Conversely, we can start with the trigonometric equation, and get back the equation in terms of Cartesian coordinates. Given a real number k, let the equation

$$\tan\theta\,\tan\psi = -k \tag{2.3}$$

hold. Taking $\tan\theta = y/x$ and $\tan\psi = y'$, (2.3) can be written as

$$yy' = -kx.$$

On integration, this gives:

$$kx^2 + y^2 = C^2 \qquad (C \in \mathbb{R}). \tag{2.4}$$

Equation (2.4) represents a family of ellipses if $k > 0$, and a family of hyperbolas if $k < 0$.

The Cartesian equation of a parabola is

$$y^2 = 4ax, \qquad (a \in \mathbb{R} : \text{parameter}). \tag{2.5}$$

Differentiating with respect to x, one obtains $2yy' = 4a$ and hence,

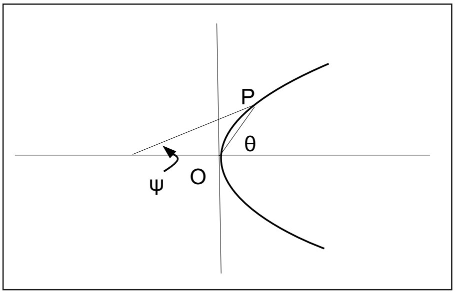
$$y^2 = 2yy'x \quad \text{or} \quad y(y - 2xy') = 0. \tag{2.6}$$

If $y = 0$, then from (2.5), $x = 0$ and hence (0, 0) is the only solution, which is a degenerate case.

If $y \neq 0$, then from (2.6) we get

$$2\tan\psi = \tan\theta. \tag{2.7}$$

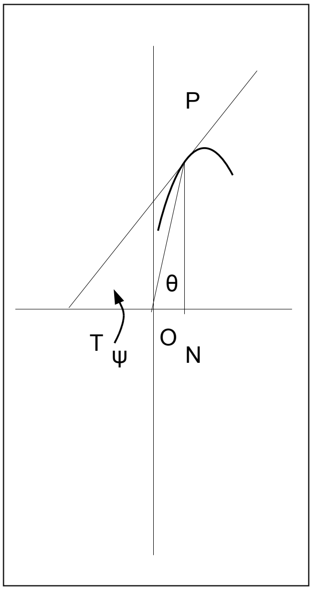
Figure 4.



Conversely, if (2.7) is true for any curve, then taking $\tan\theta = y/x$ and $\tan\psi = y'$ we get the equation $2y' = y/x$, which on integration gives a family of parabolas, given by $y^2 = Cx$, $C \in \mathbb{R}$ (*Figure* 4).

2.1 *Applications*

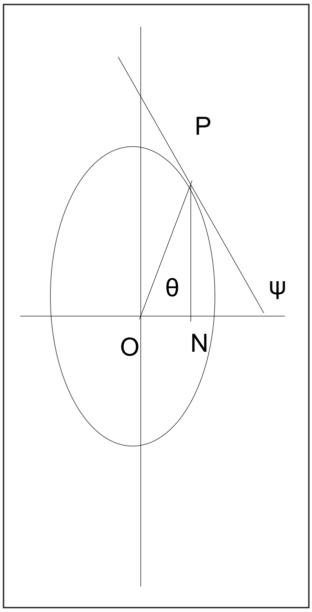
Figure 5.



Example 1. Let us find the locus of points such that the distance between the foot of the perpendicular on the x-axis from a point on the required locus and the point of intersection of the tangent to the locus at that point with the x-axis is bisected by the pole. (See *Figure* 5).

From the figure, $\frac{\mathrm{PN}}{\mathrm{ON}} = \tan\theta$ and $\tan\psi = \frac{\mathrm{PN}}{\mathrm{TN}} = \frac{\mathrm{PN}}{2\mathrm{TO}} = \frac{\mathrm{PN}}{2\mathrm{ON}} = \frac{1}{2}\tan\theta$. Therefore, $2\tan\psi = \tan\theta$. So, by (2.7), the required locus is a family of parabolas with the pole as vertex.

Figure 6.



Example 2. Let P be a point on an ellipse, N be the foot of the perpendicular from P on its major axis, and T be the point of intersection of the tangent at P with its major axis. If O is the centre of the ellipse, let us prove that $\mathrm{ON} \cdot \mathrm{OT} =$ constant (see *Figure* 6).

Taking the centre O of the ellipse as the pole and its major axis as the polar axis, we have

$$\tan\,\theta = \frac{\mathrm{PN}}{\mathrm{ON}}, \tan\,\psi = -\frac{\mathrm{PN}}{\mathrm{NT}}.$$

So

$$\tan\,\theta\tan\psi = -\frac{\mathrm{PN}^2}{\mathrm{ON}(\mathrm{OT}-\mathrm{ON})} < 0.$$

We know from Section 2 that, for an ellipse, $\tan\theta\tan\psi = -a^2$ for some real number a. Hence,

$$\text{PN}^2 = a^2(\text{ON}\cdot\text{OT} - \text{ON}^2),$$
$$\text{i.e.,}\, a^2\text{ON}\cdot\text{OT} = a^2\text{ON}^2 + \text{PN}^2.$$

For an ellipse, the right-hand side of this equation is a constant. Hence,

$$\text{ON}\cdot\text{OT} = \text{constant}.$$

3. Other Algebraic Curves

In this section, various types of families of algebraic curves are considered. Equations of these curves are written either in Cartesian coordinates (x, y) or in terms of plane polar coordinates (r, θ). In some cases, parametric equations are also considered.

3.1 *Astroid*

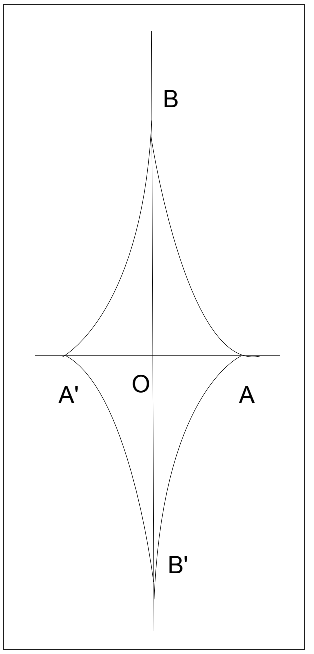


Figure 7.

In Cartesian coordinates, the equation of an astroid (*Figure* 7) is

$$x^{2/3} + y^{2/3} = a^{2/3}, \quad (a \in \mathbb{R}). \tag{3.1}$$

Differentiating both sides of (3.1) with respect to x, we get

$$y' = -\left[\frac{y}{x}\right]^{1/3}. \tag{3.2}$$

Writing $\tan\theta = y/x$ and $\tan\psi = y'$, we get from (3.2)

$$\tan\psi = -\tan^{1/3}\theta, \quad \text{or,} \quad \tan^3\psi + \tan\theta = 0. \tag{3.3}$$

Conversely, from (3.3) we can deduce (3.2). Integrating both sides, we get back the equation

$$x^{2/3} + y^{2/3} = C \quad (C \in \mathbb{R}).$$

Thus, (3.3) is the trigonometric representation of a family of astroids.

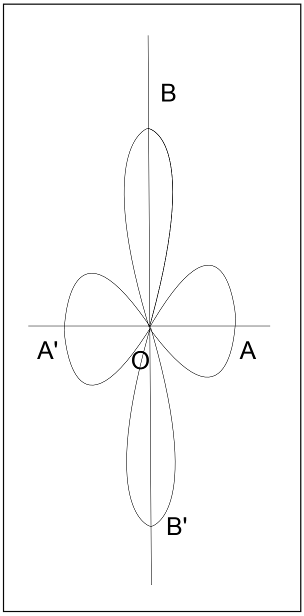


Figure 8.

3.2 *Quadrifolium*

A rose-petal curve (*Figure* 8) with four petals is called a quadrifolium. In plane polar coordinates $(r,\ \theta)$, the equation of the curve is

$$r = a\cos 2\theta. \tag{3.4}$$

Differentiating both sides with respect to θ and writing $\mathrm{d}r/\mathrm{d}\theta = \dot{r}$, we get

$$\dot{r} = -2a\sin 2\theta. \tag{3.5}$$

Differentiating the relations $r^2 = x^2 + y^2$and $\tan\theta = y/x$ with respect to x, we get

$$\dot{r} = \frac{r'}{\theta'} = r\frac{x + yy'}{xy' - y}\,.$$

Note that, $(' \equiv \frac{\mathrm{d}}{\mathrm{d}x})$.

Substituting this value of $\dot{r}$ in (3.5) we get

$$2\tan 2\theta = \frac{1 + \tan\theta\tan\psi}{\tan\theta\ - \tan\ \psi}. \tag{3.6}$$

Again, from the relations, $x = r\cos\theta$, $y = r\sin\theta$, we get

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{\dot{r}\tan\theta\ + r}{\dot{r} - r\tan\theta}.$$

Substituting this expression for $\tan\psi\ (= y')$ in (3.6), we get

$$2\tan 2\theta = -\frac{\dot{r}}{r}.$$

On integration, this gives

$$r = C\cos 2\theta.$$

This represents a family of quadrifoliums. Hence (3.6) is the trigonometric representation of this family.

3.3 *Nodal Cubic Curve*

The equation of the curve in Cartesian coordinates is

$$y^2 = x^3 + x^2. \tag{3.7}$$

Parametrically, it can be written as

$$\begin{aligned} x &= t^2 - 1, \\ y &= t^3 - t. \end{aligned} \tag{3.8}$$

Writing $\dot{x}$ and $\dot{y}$ for $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ respectively, we get

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{3t^2 - 1}{2t} = \tan\psi. \tag{3.9}$$

Also,

$$\frac{y}{x} = \frac{t(t^2-1)}{t^2-1} = t = \tan\theta\,. \tag{3.10}$$

Hence,

$$\tan\psi = \frac{3\tan^2\theta - 1}{2\tan\theta}$$

or,

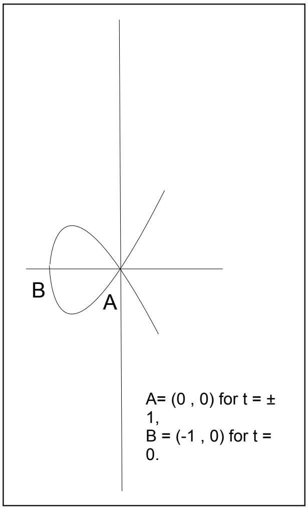
$$2\tan\theta\tan\psi = 3\tan^2\,\theta\, - 1. \tag{3.11}$$

Conversely, let (3.11) be true for a given plane curve. We get: $2y' = 3y/x - x/y$. Solving this equation, we get,

$$y^2 = Cx^3 + x^2,\ (C \in \mathbb{R} : \text{parameter}). \tag{3.12}$$

This represents a family of nodal cubic curves. Hence, (3.11) is the trigonometric equation of such a family (*Figure* 9).

Figure 9.



3.4 *Applications*

Example 3. Let us prove that the envelope of a family of ellipses for which the sum of the major and minor axes is a constant, is an astroid.

If a, b be the major and minor semi-axes of an ellipse, then, according to the hypothesis, $a + b = k$, where

$k \in \mathbb{R}$ is the given constant. Hence, the equation of the ellipse can be written as

$$F(a) \equiv \frac{x^2}{a^2} + \frac{y^2}{(k-a)^2} - 1 = 0. \tag{3.13}$$

To find the envelope, we have $\partial_a \mathrm{F} = 0$, from which we get

$$\frac{y^2}{x^2} = \left(\frac{k-a}{a}\right)^3 = \tan^2\theta. \tag{3.14}$$

Also, for the ellipse, the trigonometric equation is

$$\tan\theta\tan\psi = -\left(\frac{k-a}{a}\right)^2$$

or

$$\tan\psi = -\left(\frac{k-a}{a}\right)^{1/2} = -\tan^{1/3}\theta$$

or

$$\tan^3\psi + \tan\theta = 0. \tag{3.15}$$

$\tan^3\psi + \tan\theta = 0$ is the trigonometric equation of an astroid.

This is the trigonometric equation of an astroid.

4. Trigonometric Equations

In this section, a few simple trigonometric equations, which involve two angles, namely, θ and ψ are considered. We try to find out if there exists a family of plane curves corresponding to the given equation. Throughout, consider θ as the vectorial angle of a point on the curve, if it exists, and ψ as the angle of inclination of the tangent to the curve (if the tangent exists) at the given point to the x-axis (see *Figure* 1).

(a) Consider the equation

$$k\tan\theta + \tan\psi = 0, \quad k \in \mathbb{R}. \tag{4.1}$$

If there exists any curve satisfying this equation, then (4.1) yields: $ky/x + y' = 0$. Solving this, we get

$$y = Cx^{-k}, \qquad (C \in \mathbb{R} : \text{parameter}). \tag{4.2}$$

For different values of k, different families of curves are obtained:

(i) $k = -2, \quad y = Cx^2$: a family of parabolas having the same axis.

(ii) $k = -1, \quad y = Cx$: a family of straight lines passing through the origin.

(iii) $k = 0, \quad y = C$: a line parallel to the x-axis.

(iv) $k = 1, \quad xy = C$: a family of hyperbolas.

(v) $k = 2, \quad x^2y = C$: a family of cubic curves,

and so on.

(b) Consider the mixed equation (involving both Cartesian coordinates and the trigonometric angles, θ and ψ)

$$x \tan\psi + y \tan\theta = 0. \tag{4.3}$$

If there exists a curve satisfying this equation, then (4.3) can be written as

$$\frac{y'}{y^2} + \frac{1}{x^2} = 0$$

which gives

$$x + y + kxy = 0, \qquad k \in \mathbb{R}. \tag{4.4}$$

If $k \neq 0$, then relation (4.4) represents a family of hyperbolas.

If $k = 0$, then (4.4) reduces to $x+y = 0$, which represents a straight line passing through the origin. This is a degenerate conic.

Similarly, the equation $y \tan\psi + x \tan\theta = 0$ represents either the x-axis, $y = 0$, or the family of lines, $x + y = 0$.

(c) Finally, let $\theta + \psi =$ constant. Then,

$$\tan(\theta + \psi) = \frac{\tan\theta + \tan\psi}{1 - \tan\,\theta \tan\,\psi} = \text{constant } = k, \text{say.} \tag{4.5}$$

We get

$$k(x - yy') = y + xy'\,. \tag{4.6}$$

Integrating, we get

$$k(x^2 - y^2) = xy + C\,, \qquad C \in \mathbb{R}. \tag{4.7}$$

This represents a family of hyperbolas, unless $C = 0$, when it represents a pair of intersecting lines through the origin.

5. Remarks

The main idea of this article is to highlight the fact that trigonometric equations are very convenient for characterizing certain one-parameter family of plane curves: the trigonometric equation under consideration represents the geometrical property shared by all members of that one-parameter family of curves.

Conversely, not all trigonometric equations represent families of plane curves.

The second aspect is to highlight that, in some cases, the trigonometric equation is quicker and easier in achieving the desired characterizations of the families of plane curves than its equivalent Cartesian or plane polar representation.

Finally, it is to be noted that there is ample scope for dealing with many more trigonometric relations. The extension of this idea to three-dimensional space may also be explored.

Suggested Reading

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[2] A Pogorelov, *Geometry*, Mir Publishers, Moscow, 1987.

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