Realm of Matrices

Exponential and Logarithm Functions

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In this article, we discuss the exponential and the logarithmic functions in the realm of matrices. These notions are very useful in the mathematical and the physical sciences [1,2]. We discuss some important results including the connections established between skew-symmetric and orthogonal matrices, etc., through the exponential map.

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1. Introduction

The term 'matrix' was coined by Sylvester in 1850. Cardano, Leibniz, Seki, Cayley, Jordan, Gauss, Cramer and others have made deep contributions to matrix theory. The theory of matrices is a fundamental tool widely used in different branches of science and engineering such as classical mechanics, optics, electromagnetism, quantum mechanics, motion of rigid bodies, astrophysics, probability theory, and computer graphics [3–5]. The standard way that matrix theory gets applied is by its role as a representation of linear transformations and in finding solutions to a system of linear equations [6]. Matrix algebra describes not only the study of linear transformations and operators, but it also gives an insight into the geometry of linear transformations [7]. Matrix calculus generalizes the classical analytical notions like derivatives to higher dimensions [8]. Also, infinite matrices (which may have an infinite number of rows or columns) occur in planetary theory and atomic theory. Further, the classification of matrices into different types such as skew-symmetric, orthogonal, nilpotent, or unipotent matrices, is essential in dealing with complicated practical problems. In this article, we will discuss the method to compute the exponential of any arbitrary real or com-

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plex matrix, and discuss some of their important properties [9,10].

2. Jordan Form of Matrices

A Jordan block – named in honour of Camille Jordan – is a matrix of the form

$$J = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

Every Jordan block that is described by its dimension n and its eigenvalue λ, is denoted by $J_{\lambda,n}$.

DEFINITION 2.1

If M_n denotes the set of all $n \times n$ complex matrices, then a matrix $A \in M_n$ of the form

$$A = \begin{pmatrix} A_{11} & & & 0 \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{kk} \end{pmatrix}$$

in which $A_{ii} \in M_{n_i}, i = 1, 2, \dots, k$, and $n_1 + n_2 + \dots + n_k = n$, is called a block diagonal. Notationally, such a matrix is often indicated as $A = A_{11} \oplus A_{22} \oplus \dots \oplus A_{kk}$; this is called the direct sum of the matrices $A_{11}, A_{22}, \dots, A_{kk}$ [7].

A block diagonal matrix whose blocks are Jordan blocks, is called a Jordan matrix, denoted by using either $\oplus$ or diag symbol.

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The $(m+s+p) \times (m+s+p)$ block diagonal square matrix, having first, second, and third diagonal blocks $J_{a,m}$, $J_{b,s}$ and $J_{c,p}$ is compactly indicated as $J_{a,m} \oplus J_{b,s} \oplus J_{c,p}$ or, $\text{diag}(J_{a,m}, J_{b,s}, J_{c,p})$ respectively [4,7]. For example, the square matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

is a 10×10 Jordan matrix with a 3×3 block with eigenvalue 0, two 2×2 blocks with imaginary unit i and a 3×3 block with eigenvalue 5. Its Jordan block structure can be expressed as either $J_{0,3} \oplus J_{i,2} \oplus J_{i,2} \oplus J_{5,3}$ or, $\mathrm{diag}(J_{0,3}, J_{i,2}, J_{i,2}, J_{5,3})$.

3. Nilpotent and Unipotent Matrices

DEFINITION 3.1

A square matrix X is said to be nilpotent if $X^r = 0$ for some positive integer r. The least such positive integer is called the index (or, degree) of nilpotency. If X is an $n \times n$ nilpotent matrix, then $X^m = 0$ for all $m \geq n$ [9].

For example, the 2×2 matrix $A = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$ is nilpotent of degree 2, since $A^2 = 0$. In general, any triangular matrix with zeros along the main diagonal is nilpotent.

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For example, the 4×4 matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is nilpotent of degree 4 as $A^4 = 0$ and $A^3 \neq 0$. In the above examples, several entries are zero. However, this may not be so in a typical nilpotent matrix.

For instance, the 3×3 matrix

$$A = \begin{pmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{pmatrix}$$

squares to zero, i.e., $A^2 = 0$, though the matrix has no zero entries.

For $A \in M_n$, the following characterization may be worth mentioning:

- Matrix A is nilpotent of degree $r \leq n$ i.e., $A^r = 0$.
- The characteristic polynomial $\chi_A(\lambda) = \det(\lambda I_n - A)$ of A is λ^n.
- The minimal polynomial for A is λ^r.
- $\text{tr}(A^r) = 0$ for all $r > 0$, i.e., the sum of all the diagonal entries of A^r vanishes.
- The only (complex) eigenvalue of A is 0.

Further, from the above, the following observations can be added:

- The degree of an $n \times n$ nilpotent matrix is always less than or equal to n.
- The determinant and trace of a nilpotent matrix are always zero.
- The only nilpotent diagonalizable matrix is the zero matrix.

3.2 *Canonical Nilpotent Matrix*

We consider the $n \times n$ shift matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

The only nilpotent diagonalizable matrix is the zero matrix.

which has ones along the super diagonal and zeros at other places. As a linear transformation, this shift matrix shifts the components of a vector one slot to the left: $S(a_1, a_2, \ldots, a_n) = (a_2, a_3, \ldots, a_n, 0)$. As, $A^n = 0 \neq A^{n-1}$, this matrix A is nilpotent of degree n and is called the canonical nilpotent matrix. Further, if A is any nilpotent matrix, then A is similar to a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & O & \ldots & O \\ O & A_2 & \ldots & O \\ \vdots & \vdots & \ddots & O \\ O & O & \ldots & A_r \end{pmatrix},$$

where each of the blocks $A_1, A_2, \ldots, A_r$ is a shift matrix (possibly of different sizes). The above theorem is a special case of the Jordan canonical form of matrices. For example, any non-zero, nilpotent, 2-by-2 matrix A is similar to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. That is, if A is any non-zero nilpotent matrix, then there exists a basis $\{b_1, b_2\}$ such that $Ab_1 = O$ and $Ab_2 = b_1$. For example, if $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $b_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $b_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, then $Ab_1 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and $Ab_2 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = b_1$.

3.3 *Properties*

(i) If A is a nilpotent matrix, then $I + A$ is *invertible.* Moreover, $(I + A)^{-1} = I - A + A^2 - A^3 + \cdots + (-1)^{n-1}A^{n-1}$, where the degree of A is n.

(ii) If A is nilpotent then $\det(I+A) = 1$. For example, if $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then, $A^2 = O$ and $\det(I + A) = 1$. Conversely, if A is a matrix and $\det(I + tA) = 1$ for all values of scalar t then A is nilpotent.

(iii) Every singular matrix can be expressed as a product of nilpotent matrices.

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DEFINITION 3.4.

An $n \times n$ matrix A is said to be unipotent if the matrix $A - I$ is nilpotent. The degree of nilpotency of $A - I$ is also called the degree of unipotency of A.

An $n \times n$ matrix A is said to be unipotent if the matrix $A - I$ is nilpotent.

For example,

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 6 & -3 & 2 \\ 15 & -8 & 6 \\ 10 & -6 & 5 \end{pmatrix}$$

are unipotent matrices of degree 2, 4 and 2 respectively because $(A - I)^2 = O$, $(B - I)^4 = O$ and $(C - I)^2 = O$.

We know that every complex matrix X is similar to an upper triangular matrix. Thus, there exists a non-singular matrix P such that $X = PAP^{-1}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

Therefore, the characteristic polynomial of X is $(\lambda - a_{11})(\lambda - a_{22}) \dots (\lambda - a_{nn})$, as similar matrices have the same characteristic polynomial. Then two cases may arise:

Case I. The eigenvalues $a_{11}, a_{22}, \dots, a_{nn}$ are all distinct.

Case II. Not all of $a_{11}, a_{22}, \dots, a_{nn}$ are distinct.

For example, consider the matrix $A = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 4 \end{smallmatrix}\right)$.

Then $A-\lambda I = \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix}$ and $\det(A-\lambda I) = (\lambda-5)(\lambda-2)$. The determinant vanishes if $\lambda = 5$ or 2 which are the distinct eigenvalues of A. Now to find the eigenvectors of the matrix equation $AX = \lambda X$, we solve the two systems of linear equations $(A-5I)X = 0$ and $(A-2I)X = 0$ where from the eigen vectors are obtained as $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

These eigenvectors form a basis $B = (v_1, v_2)$ of $\mathbb{R}^2$ and the matrix relating the standard basis E to the basis B is $P = (B)^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1} = -\frac{1}{3}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}$ and $PAP^{-1} = A'$ is diagonal: $A' = -\frac{1}{3}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$, which is the Jordan canonical form of A, and its characteristic polynomial is $(\lambda-5)(\lambda-2)$. The two distinct eigenvalues are 5 and 2.

4. Exponential of a Matrix

Recall that the exponential function $e^z = \sum_{n\geq 0} \frac{z^n}{n!}$ is a convergent series for each $z \in \mathbb{C}$. Before we explain the meaning of the above infinite series when z is replaced by a general $n \times n$ matrix, we describe it for the case of nilpotent matrices. If X is a nilpotent matrix of degree r, then by definition $X^r = 0$, $r \leq n$, so that

$$e^X = \sum_{n\geq 0} \frac{X^n}{n!} = I + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots + \frac{X^{r-1}}{(r-1)!}$$

is a polynomial in X of degree $(r-1)$.
For example, consider the 4×4 matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By direct multiplication

$$A^2 = \begin{pmatrix} 0 & 0 & 2 & 11 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $A^4 = 0$; therefore, A is a nilpotent matrix of degree 4. The exponential series for this matrix A reduces to $e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3$ and by matrix addition, it yields a 4×4 matrix

$$e^A = \begin{pmatrix} 1 & 1 & 3 & \frac{67}{6} \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is invertible since $\det(e^A) = 1$.

To explain the meaning of the power series for a general matrix, define the norm of any $A \in M_n$ to be

$$||A|| = \mathrm{Max}_{i,j}|a_{ij}|.$$

Note that it is easy to check

$$||AB|| \leq n||A||||B||.$$

The norm gives a notion of distance between matrices viz., the distance between A and B is the norm of $A - B$. With this definition, it follows that

$$||\sum_{r=0}^{k} \frac{A^r}{r!}|| \leq 1 + \frac{e^{n||A||} - 1}{n} \ \forall\ k.$$

Therefore, in M_n, the sequence of matrices $\sum_{r=0}^{k} \frac{A^r}{r!}$ converges to a matrix as $k \to \infty$; this matrix is denoted by e^A.

Here are some properties of the exponential of a matrix:

$\det(e^A) = e^{tr(A)}$ for any $A \in M_n$.

- $e^{O_n} = I_n$, where O_n is the $n \times n$ zero matrix.
- $Pe^AP^{-1} = e^{PAP^{-1}}$.
- If B is upper triangular, then e^B is upper triangular with diagonal entries $e^{b_{ii}}$. In particular, since every matrix is similar to an upper triangular matrix, we have

$$\det(e^A) = e^{tr(A)}$$

for any $A \in M_n$.

5. Logarithm of Unipotent and other Matrices

Recall that $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for $|x| < 1$. If a matrix A is nilpotent of degree r then $A^r = O$ for a positive constant r, and it makes sense to define

$$\log(I + A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \ldots + (-1)^{r-2}\frac{A^{r-1}}{(r-1)}.$$

On the other hand, if the matrix A is unipotent, then $A-I$ is nilpotent so that $(A-I)^r = O$ for some positive constant r. Let $N = A - I$. Therefore,

$$\log A = (A - I) - \frac{(A-I)^2}{2} + \frac{(A-I)^3}{3} - \ldots + (-1)^{r-2}\frac{(A-I)^{r-1}}{r-1}.$$

For example, consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that $A-I, (A-I)^2, (A-I)^3$ are non-zero matrices and $(A-I)^4 = O$. Then, $\log A = (A - I) - \frac{(A-I)^2}{2} + \frac{(A-I)^3}{3}$, becomes

$$\log A = \begin{pmatrix} 0 & 1 & 1 & \frac{11}{6} \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is the logarithm of the unipotent matrix A of degree 4.

More generally, we may define:

DEFINITION 5.1.

The logarithm of a square matrix $I + A$ with $||A|| < 1$ is defined by [10]

$$\log(I + A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \dots .$$

This series is convergent for $||A|| < 1$, and hence $\log(I + A)$ is a well-defined continuous function in this neighborhood of I. The fundamental property of the matrix logarithm is the same as that of ordinary logarithm; it is the inverse of exponential function [10], i.e., $\log A = B$ implies $e^B = A$.

The fundamental property of the matrix logarithm is the same as that of ordinary logarithm; it is the inverse of exponential function.

Lemma 5.2. *If an $n \times n$ matrix A is unipotent, then $\log A$ is nilpotent.*

Proof. We know that for any $n \times n$ matrix A,

$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - I)^m}{m} \qquad (1)$$

whenever the series converges.

Since A is unipotent, then $A - I$ is nilpotent. Let the degree of nilpotency of $(A - I)$ be r. Then $(A - I)^m = O$ for all $m \geq r$. So the series (1) becomes

$$\log A = \sum_{m=1}^{r-1} (-1)^{m+1} \frac{(A - I)^m}{m}, \qquad (2)$$

i.e., the series terminates. So $\log A$ is defined whenever A is unipotent.

Let $A-I=N$. Then $A=I+N$ and $N^r=(A-I)^r=O$. Therefore from (2),

$$\log A = N - \frac{N^2}{2} + \frac{N^3}{3} - \cdots (-1)^r \frac{N^{r-1}}{r-1},$$

and so

$$\begin{aligned} (\log A)^r &= \left(N - \frac{N^2}{2} + \frac{N^3}{3} - \cdots (-1)^r \frac{N^{r-1}}{r-1}\right)^r \\ &= N^r \left(I - \frac{N}{2} + \frac{N^2}{3} - \cdots (-1)^r \frac{N^{r-2}}{r-1}\right)^r \\ &= O. \end{aligned}$$

Therefore $\log A$ is nilpotent of degree r.

PROPOSITION 5.3. *If a matrix A is unipotent, then* $\exp(\log A) = A$.

Proof. Since A is unipotent, then $A = I + N$, where N is nilpotent. As such from the above lemma, $\log A$ is defined. Let the degree of nilpotency of N be r, that is, $N^m = O$ for all $m \geq r$. Therefore,

$\exp(\log A)$

$$\begin{aligned} &= \exp\log(I+N) \\ &= \exp\left(N - \frac{N^2}{2} + \frac{N^3}{3} - \cdots + (-1)^r \frac{N^{r-1}}{r-1}\right) \\ &= I + \left(N - \frac{N^2}{2} + \frac{N^3}{3} - \cdots + (-1)^r \frac{N^{r-1}}{r-1}\right) \\ &+\frac{1}{2!}\left(N - \frac{N^2}{2} + \frac{N^3}{3} - \cdots + (-1)^r \frac{N^{r-1}}{r-1}\right)^2 + \cdots \\ &+\frac{1}{(r-1)!}\left(N - \frac{N^2}{2} + \frac{N^3}{3} - \cdots \right. \\ &\left. +(-1)^r \frac{N^{r-1}}{r-1}\right)^{r-1} = I + N, \end{aligned}$$

If a matrix A is unipotent, then exp(log *A*) = *A.*

as the coefficients of $N^2, N^3, \ldots, N^{r-1}$ are all zero.

Hence, $\exp(\log A) = I + N = A$.

PROPOSITION 5.4 *It may be shown that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \log(I + At) = A(I + At)^{-1}.$$

Proof. By definition of logarithmic series,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \log(I + At) &= \frac{\mathrm{d}}{\mathrm{d}t} \left[At - \frac{(At)^2}{2} + \frac{(At)^3}{3} - \ldots \right] \\ &= A - A^2 t + A^3 t^2 - \ldots \\ &= A \left(I - (At) + (At)^2 - \ldots \right) \\ &= A(I + At)^{-1}. \end{aligned}$$

6. Properties of Exponential Map on Matrices

We have already mentioned some properties (the first three below) of the exponential map on matrices. Let us recall some more.

- $e^{O_n} = I_n$, where O_n is the $n \times n$ zero matrix.
- $Pe^A P^{-1} = e^{PAP^{-1}}$.
- If B is upper triangular, then e^B is upper triangular with diagonal entries $e^{b_{ii}}$. In particular, since every matrix is similar to an upper triangular matrix, we have

  $$\det(e^A) = e^{tr(A)}$$

  for any $A \in M_n$.
- If $A, B \in M_n$ commute (that is, $AB = BA$), then $e^{A+B} = e^A e^B$.

If A is any $n \times n$ complex matrix, then the exponential e^A is invertible and its inverse is e^{-A}.

- If A is any $n \times n$ complex matrix, then the exponential e^A is invertible and its inverse is e^{-A}.

We point out the following examples for computing the exponential map:

(1) Consider a triangular 2×2 matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$. Then

$$A^2 = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}, A^3 = \begin{pmatrix} 1 & 7 \\ 0 & 8 \end{pmatrix}, A^4 = \begin{pmatrix} 1 & 15 \\ 0 & 16 \end{pmatrix}, \cdots ,$$

so that

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \ldots \\ &= \begin{pmatrix} e & e^2 - e \\ 0 & e^2 \end{pmatrix}. \end{aligned}$$

(2) The above matrix e^A is a non-singular matrix and its inverse is computed as

$$e^{-A} = (e^A)^{-1} = \begin{pmatrix} e^{-1} & e^{-2} - e^{-1} \\ 0 & e^{-2} \end{pmatrix}.$$

(3) Consider a 2×2 real matrix $A = \left(\begin{smallmatrix} 2 & 3 \\ 0 & 2 \end{smallmatrix}\right)$ which may be expressed as $A = 2I + B$, where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 3 \\ 0 & 0 \end{smallmatrix}\right)$. Since $2I$ commutes with B, we can write $e^A = e^{2I+B} = e^{2I}e^B$. From the exponential series, we find $e^{2I} = e^2 I$, and $e^B = I + B = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)$ as $B^2 = 0$. Therefore,

$$e^A = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^2 & 3e^2 \\ 0 & e^2 \end{pmatrix}$$

which is a non-singular matrix with inverse $e^{-A} = \left(\begin{smallmatrix} e^{-2} & -3e^{-2} \\ 0 & e^{-2} \end{smallmatrix}\right)$.

(4) Consider a 2×2 diagonal matrix $A = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$.

Then

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \ldots = \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}.$$

Its inverse is computed as $e^{-A} = \left(\begin{smallmatrix} e^{-d_1} & 0 \\ 0 & e^{-d_2} \end{smallmatrix}\right)$.

7. Relation of Skew-symmetric Matrices to Orthogonal Matrices

DEFINITION 7.1

An $n \times n$ matrix A satisfying $A + A^t = 0$, is called a skew-symmetric matrix. That is, its transpose is equal to the negative of itself, i.e., $A^t = -A$.

The name skew-symmetric comes because the reflection of each entry about the diagonal is the negative of that entry. In particular, all leading diagonal elements are zero, i.e., $tr(A) = 0$.

DEFINITION 7.2

An $n \times n$ matrix A satisfying $AA^t = I$, is called an orthogonal matrix. That is, its transpose equals its inverse, $A^t = A^{-1}$, [5].

For examples, the following are skew-symmetric matrices:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -h & g \\ h & 0 & -f \\ -g & f & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -1 & 2 & -3 \\ 1 & 0 & -4 & 5 \\ -2 & 4 & 0 & -6 \\ 3 & -5 & 6 & 0 \end{pmatrix}.$$

Some examples of orthogonal matrices are:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The addition formula of the exponential function is $e^{A+B} = e^A e^B$, when $AB = BA$. If X is skew-symmetric, then the matrices X and X^t commute, as $X + X^t = O$. That is, $XX^t = X(-X) = (-X)X = X^t X$, where X^t is the transpose of X. Thus, it follows from the above property of exponential function that $e^X e^{X^t} = e^{X+X^t} = e^O = I$, where O and I are zero and identity matrices. Also, $e^{X^t} = (e^X)^t$. Therefore, $I = e^X e^{X^t} = e^X (e^X)^t$. That is, if X is skew-symmetric, then e^X is an orthogonal matrix. Further, e^X has determinant 1, as $\det(e^X) = e^{tr(x)} = e^0 = 1$.

Finally, we leave as an exercise the connection between skew-hermitian matrices and unitary matrices. The exponential of a matrix plays a very important role in the theory of Lie groups.

Suggested Reading

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