A Short Proof of Euler's Inequality $R \geq 2r$

Theorem. *Let* Δ ABC *be an arbitrary triangle with circumradius* R *and inradius* r. *Then* $R \geq 2r$ *with equality holding if and only if* ΔABC *is equilateral.*

This was first published by Euler in 1765. Since then several proofs have followed, some geometric and some algebraic. We will use relations between inradius and exradii (r_a, r_b, r_c) to prove the inequality. The following are standard identities, and their proofs can be obtained from any book on trigonometry.

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}; \tag{1}$$

$$r_a + r_b + r_c - r = 4R. \tag{2}$$

It is known that for any three positive real numbers x, y, z, one has

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9,$$

with equality holding if and only if $x = y = z$.

Hence one has from (1) and (2) above

$$\frac{4R+r}{r} = (r_a + r_b + r_c)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) \geq 9,$$

which gives the desired inequality $R \geq 2r$. Equality holds if and only if $r_a = r_b = r_c$, that is, if and only if

$$\frac{\Delta}{s-a} = \frac{\Delta}{s-b} = \frac{\Delta}{s-c} \Leftrightarrow a = b = c.$$

(Here Δ is the area and s is the semiperimeter of the triangle.)

Samer Seraj, samer.seraj@mail.utoronto.ca