Odd if it isn't an Even Fit!

Lighter Side of Tiling

B Sury

There was a chap who could tile a square
whom I was perfectly willing to hire.
"***Used triangles – all of areas same,***
and needed but eleven for this game",
he said, and I knew he was a liar!

1. Tiling Squares by Triangles of Given Area

Try to cut a square into finitely many triangles (possibly of different shapes) of equal area. You would find that – no matter what the shapes are – the number of triangles is always even. Here is an example (*Figure* 1).

There is some interesting history behind the discovery of the above fact. In 1965, Fred Richman from the University of New Mexico had decided to pose this in an examination in the master's programme. He had observed this in some cases but when he tried to prove it in general prior to posing it in the exam, he was unsuccessful. So, the problem was not posed in the exam. His colleague and bridge partner John Thomas tried for a long time and finally came up with a proof that it is impossible to break the unit square with corners at $(0,0),(1,0),(0,1),(1,1)$ into an odd number of triangles when the vertices of all the triangles have rational coordinates with odd denominators. He sent the paper to the *Mathematics Magazine* where the referee thought the result may be fairly easy (but could not find a proof himself) and perhaps known (but could not find a reference to it). On the referee's suggestion, Richman and Thomas posed this as a problem [2] in the *American Mathematical Monthly* which nobody could solve. Subsequently, Thomas's paper appeared in the *Mathematics Magazine* [4] in 1968. Finally, in 1970, Paul Monsky

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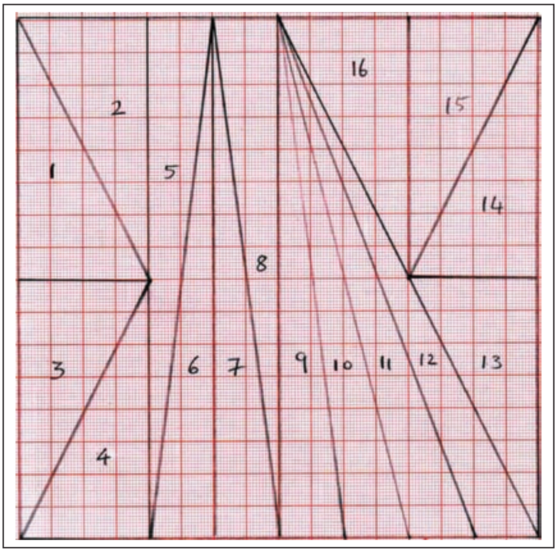
Figure 1.



proved the complete version in a paper [1] in the *American Mathematical Monthly* removing the restriction imposed in Thomas's paper.

We shall discuss Monsky's proof of this beautiful fact presently. Amazingly, it uses some nontrivial mathematical objects called 2-adic valuations.

In fact, one may consider generalizations of squares like cubes and hypercubes of higher dimensions. If an n-dimensional cube is cut into simplices (generalizations of triangles like tetrahedra, etc., in higher dimensions) of equal volumes, it turns out that the number of simplices must be a multiple of n!

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A region like the interior of a square is said to be 'tiled' by triangles if the square can be broken into triangular pieces. There is also a generalization of the result on tiling by triangles of squares to the (so-called) polyominos. Polyominos are just unions of unit squares. Connected subsets of the square lattice tiling of the plane are called special polyominos. That is, they have standard edges – edges of the unit squares are parallel to the co-ordinate axes. The generalization alluded to is due to S K Stein [3] and asserts:

Consider a special polyomino which is the union of an odd number of unit squares. If this polyomino is a union of triangles of equal areas, then the number of triangles is even.

We discuss the proof of this statement after analysing the solution by Paul Monsky of the first problem we started with. It can be noticed that the proof uses crucially that the number of unit squares in the polyomino is odd. Interestingly, this question is still unanswered when the number of unit squares in the polyomino is even.

Stein's question on the parity of the number of triangles of equal areas which can tile a special polyomino is still unanswered if the polyomino is made up from an even number of unit squares.

The proof of Monsky as well as Stein's result above use the so-called '2-adic valuation function'. The 2-adic valuation is a function from the set of non-zero rational numbers to the set of integers; it simply counts the power of 2 dividing any integer or, more generally, any rational number. What is meant by the power of 2 dividing a rational number? Writing any non-zero rational number as p/q with p, q having no common factors, one may look at the power of 2 dividing p or q. If p, q are both odd, this is simply 0. If p is even, then the 2-adic valuation of p/q is defined to be the power of 2 dividing p. If q is even, then the 2-adic valuation of p/q is defined to be the negative of the power of 2 dividing q, Formally, we write:

$$\phi : \mathbf{Q}^* \to \mathbf{Z}$$

defined by $\phi(\frac{2^a b}{c}) = a$, where b, c are odd; define also $\phi(0) = \infty$ so that ϕ is defined for all rational numbers. We keep in mind that 0 has a larger 2-adic value than any other rational number.

Colour a point $(x, y) \in \mathbf{Q} \times \mathbf{Q}$ by the colour:

$$\begin{gathered}\text{red, if } \phi(x), \phi(y) > 0,\\ \text{blue, if } \phi(x) \leq 0; \phi(x) \leq \phi(y),\\ \text{green, if } \phi(x) > \phi(y) \text{ and } \phi(y) \leq 0.\end{gathered}$$

In this manner, all points of $\mathbf{Q}\times\mathbf{Q}$ are coloured by these three colours.

For example:

$(2,0)$ is red, $(1,3)$ is blue and $(1,1/2)$ is green. Also,

$(0,0)$ is red while $(1,0)$ is blue and $(0,1)$ is green.

Now, we proceed to assert something which is easy to believe but not that easy to prove. The assertion is that it is possible to extend the above function to a function on the whole of real numbers (but the values can be non-integers). In further discussion, we assume without further ado, the existence of an extension

$$\phi : \mathbf{R} \to \mathbf{R}$$

which satisfies:

ϕ restricts to the 2-adic valuation on $\mathbf{Q}$;
$\phi(xy) = \phi(x) + \phi(y)$;
$\phi(x+y) \geq min(\phi(x), \phi(y))$.

It is important to have such an extension because we would really like to colour **all** *points in the square.*

For instance, as $\phi(3/4) = \phi(2^{-2}3) = -2$, the second property above implies that

$$\phi(3/4) = 2\phi(\sqrt{3}/2) = -2.$$

Hence $\phi(\sqrt{3}/2) = -1$.

Let us start Monsky's proof by first considering the unit square with a left lower corner at $(0,0)$ and making a few easy observations:

(i) *If a point 'a' is red, then points x and $x+a$ have the same colour.*

(ii) *On any line, there are, at the most, two colours.*

(iii) *The boundary of the square has an odd number of segments which have a red end and a blue end.*

(iv) *If a triangle is not 'complete', (that is, has vertices only of one or two colours), then it has* 0 *or* 2 *red-blue edges.*

Proof. Recall:

$$\phi(xy) = \phi(x) + \phi(y)\,,$$

$$\phi(x + y) \geq min(\phi(x), \phi(y))\,.$$

In particular, if $\phi(x) > \phi(y)$, then $\phi(x + y) = \phi(y)$. In particular, if (x, y) is blue, then $\phi(x) \leq \phi(y)$, and so, $\phi(y/x) \geq 0$ and, if (x, y) is green, then $\phi(x) > \phi(y)$ and so, $\phi(y/x) < 0$.

Let us prove (i) now.

As a is red, its co-ordinates have positive ϕ and it is easy to check in each of the three cases of colouring for a point x that x and $x + a$ have the same colour.

For (ii), without loss of generality, we may assume that the line passes through the origin. But two other points $(x_i, y_i); i = 1, 2$ on the line $y = tx$ have colours blue and green respectively, say.

But then, $\phi(y_1/x_1) = \phi(y_2/x_2) = \phi(t)$ is impossible as the former is ≥ 0, while the latter is < 0.

To prove (iii), note that (ii) implies that such segments on the boundary must be on the segment from $(0, 0)$ to $(1, 0)$ which are red and blue, respectively. But, this is clear.

The proof of (iv) is completely clear by considering the various possibilities RRB,RBB,RGG,RRG,BBG,BGG.

Now, we are ready to prove: *Let a square be tiled by* n *triangles of equal areas. Then,* n *is even.*

Proof(Monsky). Counting the red-blue edges on the square, we are counting the interior edges twice and the boundary edges once. Thus, (iii) above would be contradicted unless there is a complete triangle. But then a complete triangle has area A with $\phi(A) < 0$. Let us check this now.

Firstly, note that the triangle can be moved so that the vertices are at $(0,0),(a,b)$ and (c,d) where (a,b) is blue and (c,d), is green. Thus, the area is $(ad-bc)/2$.

As (a,b) is blue, $\phi(a) \leq \phi(b)$ and as (c,d) is green, $\phi(c) > \phi(d)$. Therefore, $\phi(ad) = \phi(a)+\phi(d) < \phi(b)+\phi(c) = \phi(bc)$, which gives $\phi(ad-bc) = \phi(ad) = \phi(a)+\phi(d) \leq 0$.

Hence, $\phi(A) = \phi((ad-bc)/2) \leq -1$. So, if there are n triangles, then $\phi(A) = \phi(1/n) < 0$; that is, n is even. This completes Monsky's wonderful proof.

Let us now prove the more general version on polyominos mentioned above: *Consider a polyomino which is the union of an odd number of unit squares. If it is tiled by triangles of equal areas, then the number of triangles is even.*

Proof. It can be seen that if a line segment made up of segments parallel to the axes has a blue end and a green end, then each of the individual segments has ends only coloured blue or green and, an odd number of them have both colours as ends.

The key observation is: *If a polyomino made up of standard squares as above is made up of n triangles of equal areas and, if an odd number of standard edges on its boundary have ends coloured blue and green, then $\phi(2A) \leq \phi(n)$, where A is the area of the polyomino.*

The proof of this, in turn, depends on the following fact: *Let (x_i, y_i); $i = 0,1,2$ be the vertices of a triangle T, where $(x_i, y_i) \in S_i$ with*

$$S_0 = \{(x,y) : \phi(x), \phi(y) > 0\},$$

$$S_1 = \{(x, y) : \phi(x) \leq 0, \phi(y)\},$$
$$S_2 = \{(x, y) : \phi(y) < \phi(x), \phi(y) \leq 0\}.$$

Then, ϕ (area $(T)) \leq -\phi(2)$.

A 10 x 15 rectangle cannot be tiled by copies of 1 x 6 rectangles.

Proof. As translation by $(-x_1, -y_1)$ does not change areas, and $P_i - P_0 \in S_i$ for any $P_i \in S_i$, we may assume that $(x_0, y_0) = (0, 0)$.

Then, area $(T) = \frac{1}{2}|x_1y_2 - x_2y_1|$.

Now $\phi(x_1) \leq 0, \phi(y_1)$.

Also, $\phi(y_2) \leq 0$ and $\phi(y_2) < \phi(x_2)$.
Thus, $\phi(x_1y_2) < \phi(x_2y_1)$ and $\phi(x_1y_2) \leq 0$.
Hence, $\phi($area of $T) = \phi(1/2) + \phi(x_1y_2) \leq \phi(1/2) = -\phi(2)$.

Next, we observe: *If a polyomino made up of standard squares as above is made up of n triangles of equal areas and, if an odd number of standard edges on its boundary have ends coloured blue and green, then* $\phi(2A) \leq \phi(n)$, *where* A *is the area of the polyomino*:

Look at a triangle of the dissection which has all three colours and let B denote its area. Note that points in S_0, S_1, S_2 have different colours. Now, $nB = A$ and $\phi(B) \leq -\phi(2)$; that is, $\phi(A) - \phi(n) \leq -\phi(2)$. Therefore,

$$\phi(n) \geq \phi(2A).$$

We now proceed to show that if a special polyomino, which is the union of an odd number of unit squares, can be tiled by triangles of equal areas, then the number of triangles must be even.

We note that a standard (unit) edge with a blue end and a green end must be parallel to the X-axis and lies on a line whose height is odd.

Therefore, on the border of each standard square, there is an edge with a blue end and a green end.

Edges in the interior of the polyomino are adjacent to two standard squares, whereas those on the boundary are adjacent to one standard square of the polyomino.

As there is an odd number of standard squares, the above observation applies and, implies that $\phi(2A) \leq \phi(n)$. But, $\phi(2A) \geq 1$; so n must be a multiple of 2.

2. Tiling Rectangles by Rectangles

Let us discuss tiling integer rectangles with integer rectangles now. Can we tile a rectangle of size 28×17 by rectangles of size 4×7? At least, the area of the smaller rectangle divides that of the larger one (a necessary requirement for tiling). But, in fact, we don't have a tiling. Why?

Look at each row of the big rectangle. If we have managed to tile as required, then 17 would be a positive linear combination of 4 and 7. This is impossible.

Thus, two necessary conditions for tiling an $m \times n$ rectangle with $a \times b$ rectangles are:
(I) ab divides mn and,
(II) each of m, n should be expressible as positive linear combinations of a, b.

Are these conditions sufficient for tiling?

Look at a 10×15 rectangle which we wish to tile with copies of a 1×6 rectangle. The two necessary conditions mentioned clearly hold true in this case. However, a tiling is obviously impossible. Let us see why. In fact, more generally, we claim that for copies of an $a \times b$ rectangle to tile an $m \times n$ rectangle, a third condition that is also necessary is that a must divide either m or n and b also must divide m or n.

To demonstrate this, look at a possible tiling. We may suppose $a > 1$ (if $a = b = 1$, there is nothing to prove).

We colour the unit squares of the $m \times n$ rectangle with

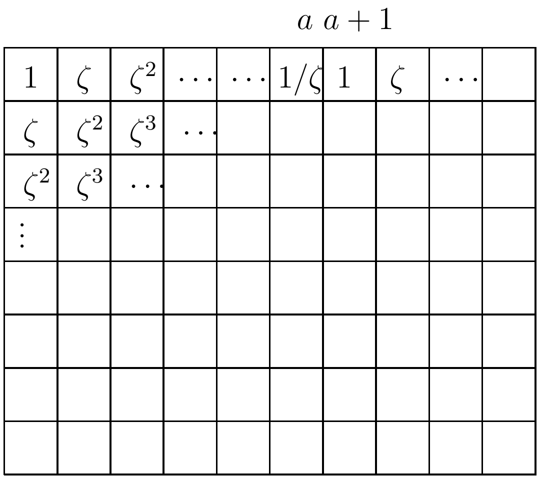

Figure 2.

the different a-th roots of unity $1, \zeta, \zeta^2, \cdots, \zeta^{a-1}$ as follows. Think of the rectangle as an $m \times n$ matrix of unit squares and colour the (i, j)-th unit square by ζ^{i+j-2}. *Figure* 2 is a suggestive tiling.

Since each tile (copy of the smaller rectangle used) contains all the a-th roots of unity exactly once, and as the sum $1 + \zeta + \cdots + \cdots \zeta^{a-1} = 0$, the sum of all the entries of the $m \times n$ rectangle must be 0.

Therefore, $\sum_{i=1}^{m} \sum_{j=1}^{n} \zeta^{i+j-2} = 0$.

But this sum is the same as $(\sum_{i=1}^{m} \zeta^{i-1})(\sum_{j=1}^{n} \zeta^{j-1}) = 0$ which means one of these two sums must be 0. But $\sum_{i=1}^{m} \zeta^{i-1} = 0$ if, and only if, $\zeta^m - 1 = 0$; that is, $a|m$. Similarly, the other sum is 0 if, and only if, $a|n$.

Thus, this condition that a divides m or n is necessary and, by the same reasoning it is necessary for tiling that b divides m or n.

Looking at the above proof, it is also easy to see how to tile when these conditions hold good. That is, we have the necessary and sufficient criterion:

PROPOSITION

An $m \times n$ rectangle can be tiled with copies of $a \times b$

Max Dehn used topological methods to prove the beautiful result that an *l* x *b* rectangle can be tiled by squares if and only if the sides are commensurable; that is, *l*/*b* is a rational number.

rectangles if, and only if, (i) ab divides mn, (ii) m and n are expressible as non-negative linear combinations of a and b, (iii) a divides m or n and b divides m or n.

This generalizes in an obvious way to any dimension and we leave it to the reader to investigate this.

We discuss now the following result for which several proofs are available.

If a rectangle is tiled by rectangles each of which has at least one of its sides integral, then the big rectangle must also have a side of integral length.

We place the co-ordinate system such that all the sides of the rectangles have sides parallel to the co-ordinate axes.

Consider the function $f(x,y) = e^{2i\pi(x+y)}$ for $(x,y) \in \mathbf{R}^2$.

For a rectangle defined by $[a,b] \times [c,d]$, we have,

$$\int\int f(x,y) = \int_a^b e^{2i\pi x}\mathrm{d}x \int_c^d e^{2i\pi y}$$

$$= \left(\frac{e^{2i\pi b} - e^{2i\pi a}}{2i\pi}\right)\left(\frac{e^{2i\pi d} - e^{2i\pi c}}{2i\pi}\right)$$

Thus, the integral of f over a rectangle is zero, if and only if, it has at least one integer side zero; hence, in case of tiling by such rectangles, the integral is zero which means that the big rectangle has an integer side.

One of the beautiful results proved by Max Dehn using methods from topology (outside our scope here) is:

A rectangle of size $l \times b$ is tileable by squares, if and only if, l/b is a rational number.

He proved more generally:

Let R be a rectangle which has at least one side of rational length. If, R is tiled by rectangles each of which has a rational ratio of length to breadth, then, all the sides of all the rectangles (including R) are of rational lengths.

Interestingly, this result of Dehn was re-proved by Brooks by associating an electrical network consisting of currents, voltages and resistances with the tiling and using well known properties of such networks. The discussions in this article would convince the reader that the subject draws from several areas of mathematics. However, we have mostly included only proofs which involve some simple algebra and basic number theory. Topological arguments require a more detailed discussion. In the next part of the article, we hope to discuss such aspects.

The result of Dehn was re-proved by Brooks by associating an electrical network and using the well-known properties of such networks.

Suggested Reading

[1] P Monsky, On dividing a square into triangles, *The American Mathematical Monthly*, Vol.77, pp.161–164, 1970.

[2] F Richman and J Thomas, Problem 5471, *The American Mathematical Monthly*, Vol.74, p.329, 1967.

[3] S K Stein and S Szabo, Algebra & Tiling, The Carus Mathematical Monographs 25, Published by the *Mathematical Association of America*, 1994.

[4] J Thomas, A dissection problem, *The Mathematics Magazine*, Vol.41, pp.187–190, 1968.

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