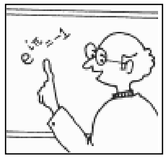
Classroom



In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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From Adiabatic Invariance to Wien's Displacement Law

An argument leading to the Wien's displacement law is presented. This simple pedagogical argument is based on the ideas of adiabatic invariance.

Max von Laue said of the 1911 Nobel Laureate, Wilhelm Wien, that his "immortal glory" was that "he led us to the very gates of quantum physics". One of the most important laws of blackbody radiation is the Wien's displacement law – a statement relating the wavelength at which the energy density is maximum, and, temperature:

$$\lambda_{\max} T = \frac{hc}{4.965 k_{\mathrm{B}}} = 2.8795 \ldots \times 10^{-3} \text{ mK}. \qquad (1)$$

The right-hand side is the Wien's constant, consisting of the Planck's constant h, the velocity of light in vacuum c, and, the Boltzmann constant, k_{B}. In standard textbooks [1], one finds the maximum of the intensity from the Planck's blackbody radiation formula by differentiating it with respect to wavelength. The motivation of writing this short article is partly historical. As we have just written, the starting point of the standard derivation of Wien's law is the radiation formula. Wien

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derived a formula for the intensity of the blackbody radiation, and one might imagine that he obtained his displacement law in the way described above. However, Wien found the formula in 1894 *but* the displacement law one year earlier, in 1893. Thus, it was possible to derive the displacement law without the knowledge of the blackbody formula. It is interesting to imagine the basis of such an argument. In the following, an argument based on adiabatic invariance is presented.

We follow Einstein in considering the radiation as a collection of harmonic oscillators. The radiation is considered to be in equilibrium at a temperature, T. For the collection of oscillators to be in a thermal equilibrium, the oscillators weakly interact in a way that even as they exchange energy among them (before coming to equilibrium), the process remains adiabatic. Each oscillator is, thus, assumed to be adiabatically perturbed. The adiabatic perturbation on an oscillator is providing the presence of the other oscillators. The frequency of this 'representative' oscillator corresponds to $\lambda_{\max}$. We know that for a slow perturbation of an oscillator, action is an adiabatic invariant [2]. That is,

A single 'representative' oscillator in the Einstein's model of radiation as a collection of harmonic oscillators is adiabatically perturbed by others. Classical action remains invariant over a time-scale of the period of the unperturbed oscillator.

$$\begin{aligned} S &= \oint p\mathrm{d}x = \oint m\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \oint 2K \ \mathrm{d}t \\ &= \langle E\rangle T = \text{constant } c_0. \end{aligned} \tag{2}$$

In this equation, the changes are so slow that the integral is over one unperturbed period of duration T. $\langle E\rangle$ is the time-average of the total energy, which is twice the time-average of kinetic energy K. This last fact can be easily proved for oscillators by considering the solution of the oscillator $x(t) = x_0\cos(\omega t+\delta)$, and calculating the time-average of kinetic energy $(mv^2/2)$ and potential energy, $m\omega^2x^2/2$.

The equation (2) is simply rewritten as $\langle E\rangle = c_0/T = c_0\nu = c_0(c/\lambda)$, where ν is the frequency of the oscillation, and the dispersion relation for an electromagnetic wave

Adiabatic invariance of action, dispersion relation of light, and equipartition theorem allows a relation between 'typical' wavelength (corresponding to maximum of the intensity distribution) and temperature.

has been employed. Since we have considered a collection of oscillators at a temperature τ, we may equate $\langle E \rangle$ to $k_B\tau$. Here, we are trying to relate the wavelength at which the intensity is maximum. With these arguments, we can now write:

$$\begin{aligned} k_B\tau &= c_0\frac{c}{\lambda} \\ \implies \lambda\tau &= c_0\frac{c}{k_B}. \end{aligned} \qquad (3)$$

This is Wien's law where the constant c_0 cannot be found as the argument is purely classical. Comparing (1) and (3), we note that c_0 is hiding the Planck's constant in it ($c_0 = \frac{h}{5}$).

This simple argument presents an interesting application of the idea of adiabatic invariance. Further, due to the fact that error in the adiabatic invariant for slow and smooth changes is exponentially small [3], the above argument gains credence.

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Suggested Reading

[1] L D Landau and E M Lifshitz, *Statistical Physics* Pergamon Press, 1958.

[2] F S Crawford, *Am. J. Phys.*, Vol.58, p.337, 1990.

[3] L D Landau and E. M. Lifshitz, *Mechanics*, II Ed. Pergamon Press, 1969.