Fractal Dimension and the Cantor Set

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The dimension of an object in its commonplace sense is the number of independent quantities needed to specify the position of points on the object. A very different definition is arrived at when we think of covering the object with copies of itself at a smaller scale and count the number of such copies needed. A virtue of this definition is that it allows us to construct objects with a 'fractional dimension'. Such objects are called fractals, and the Cantor set is one of the earliest examples of such an object.

We are familiar with the notion of *dimension* of an object, though it can be challenging to give a precise definition. Loosely speaking, it is the number of independent quantities needed to specify the positions of points on the object. Example: Consider a bead constrained to travel along a wire. The position of the bead can be specified by giving the distance along the wire from some fixed point. Hence, we say that the object is one-dimensional. The coordinate plane is two-dimensional (think of a cartesian or polar coordinate system), as is the surface of a sphere. Similarly, the space we live in is three-dimensional (or four-dimensional, if we wish to consider space-time). This notion of dimension can be made more precise (it is needed, because there are sets for which this intuitive picture is not good enough to proceed), and mathematicians refer to it as *topological dimension* (d_T for short).

Keywords
Dimension, topological dimension, Hausdorff–Besicovitch dimension, fractal dimension, fractal, Cantor set, Sierpinski triangle, Koch curve.

There is another way by which we can regard dimension, and it leads to a completely new paradigm. Consider a line segment. If we shrink it to 1/2 its size, then, we need 2 copies of the new object to cover the original object. If we shrink it to 1/3rd its size, then we need 3 copies of the new object to cover the original object.

There are objects for which the two dimensions are unequal.

Now consider a square. If we shrink it to 1/2 its size, then we need 4 copies of the new object to cover the original object. If we shrink it to 1/3rd its size, then we need 9 copies of the new object to cover the original object.

Next, consider a cube. If we shrink it to 1/2 its size, then we need 8 copies of the new object to cover the original object. If we shrink it to 1/3rd its size, then we need 27 copies of the new object to cover the original object.

What do we make of these numbers? In general, say, we have an object of such a type that it is possible to completely cover the object using smaller copies of itself, with no gaps or overlaps. Suppose that, when we reduce the scale of an object by a factor of k (i.e., we shrink it to $1/kth$ of its original size), we require N copies of the new object to cover the original one. What is the relation between N and k? For the line segment, $N = k$; for the square, $N = k^2$; and for the cube, $N = k^3$. In general, if $N = k^d$, we may refer to d as a kind of dimension of the object. This notion, when generalized suitably to apply to objects that do not have the property of self-similarity, gives the *Hausdorff–Besicovitch dimension* of the object (d_{HB} for short). The topological dimension and Hausdorff–Besicovitch dimension coincide for the line segment, the square and the cube. But there are objects for which the two do not coincide; such objects are known as *fractals*. For these, we have $d_{\mathrm{HB}} > d_{\mathrm{T}}$ (strictly). Loosely speaking, such an object occupies more space than its topological dimension suggests. The value of d_{HB} may be non-integral (which explains the name 'fractal' for such an object; hence the Hausdorff–Besicovitch dimension is also sometimes called 'fractal

A fractal occupies more space than its topological dimension would suggest.

The Cantor set was first discovered by H J S Smith.

dimension'). We now give two examples of such objects to show that this notion is not vacuous.

Cantor Set

Start with the unit interval $[0, 1]$. Delete the open middle third of the segment, leaving behind two closed segments: $[0, 1/3]$ and $[2/3, 1]$. Note that, each has length $1/3$. Repeat the same construction for each of them, namely, delete their open middle thirds. After this step, four closed intervals are left: $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$ and $[8/9, 1]$. Each of these has length $1/9$. Repeat the construction yet again; namely, delete the open middle thirds of each closed interval remaining. Continue these steps indefinitely. The construction is depicted in *Figure* 1 (with the thickness of the line segments shown exaggerated for visual clarity).

Now, examine carefully the portion of the object corresponding to the initial one-third of the original segment. You will see that it is an exact replica of the complete object, but at 1/3rd its scale. Note, moreover, that you need 2 copies of the scaled-down object to cover the object at full-scale. Hence, if the Hausdorff–Besicovitch dimension is d, then $3^d = 2$, giving $d = \log 2/\log 3 \approx 0.631$, a non-integral quantity. The topological dimension of the Cantor set may be shown to be 0.

The Cantor set was originally discovered by the British mathematician H J S Smith but studied by Cantor in quite a different way.

Figure 1. First few stages of the construction of the Cantor set.

Sierpinski Triangle

Instead of a line segment, what is the result if we work with an equilateral triangle? We do the following: Start with an equilateral triangle. Using the midpoints of its sides, divide it into four congruent equilateral triangles, each with half the scale of the original. Now remove the innermost triangle, leaving behind three such triangles. Repeat the construction with each of these triangles. And continue in this manner, indefinitely. The resulting configurations are depicted in *Figure* 2.

There is another, very different way of creating the Sierpinski triangle, using a randomized process.

To compute the Hausdorff–Besicovitch dimension d of this object, observe that, if we make a copy of the object scaled down by a factor of 2, we need 3 copies of this to cover the original triangle. Hence $2^d = 3$, giving $d = \log 3/\log 2 \approx 1.585$.

In passing we note that there is another way of constructing the Sierpinski triangle, incorporating a probabilistic element. For details, please refer to the webpage https://en.wikipedia.org/wiki/Sierpinski_triangle. See the section titled "Chaos game".

There are many well-known objects that may be constructed using iterative schemes of a similar nature.

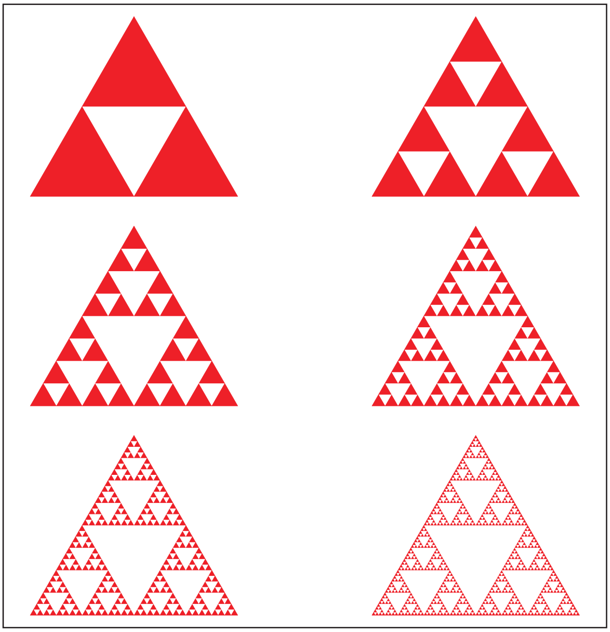

Figure 2. First few stages of the construction of the Sierpinski triangle.

There are many objects in nature for which a fractal model seems very appropriate.

More Examples

There are many well known objects that may be constructed using iterative schemes of a similar nature. Here are some that the reader may wish to explore:

(i) the *Koch curve*, also known as the snowflake curve, with fractal dimension $\log 4/\log 3 \approx 1.26$ see https://en. wikipedia. org/wiki_Koch snowflake;

(ii) the *Menger sponge*, with fractal dimension $\log 20/\log 3 \approx 2.73$ see https://en.wikipedia.org/wiki/Menger_sponge;

(iii) the *Sierpinski carpet*, with fractal dimension $\log 8/\log 3 \approx 1.89$ see https://en.wikipedia.org/wiki/Sierpinski _carpet.

The following webpage lists a large number of fractals, along with sketches and their dimensions: https://en.wikipedia.org/wiki/List_of_fractals_by_Hausdorff_dimension.

There are numerous objects in nature for which a fractal model seems particularly appropriate, e.g., broccoli with its tiny florets which seem to be miniature copies of the original (see http://www.fourmilab.ch/images/Romanesco/), capillaries within our bodies, river networks, and coastlines.

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Suggested Reading

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