How did Cantor Discover Set Theory and Topology?

S M Srivastava

In order to solve a precise problem on trigonometric series, "Can a function have more than one representation by a trigonometric series?", the great German mathematician Georg Cantor created set theory and laid the foundations of the theory of real numbers. This had a profound impact on mathematics. In this article, we narrate this fascinating story.

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1. Introduction

In this article, we recall the story of the discovery of set theory and point set topology by Georg Cantor (1845–1918). This discovery had a profound impact on mathematics. Today, it is hardly possible to do mathematics without the language and some basic results of set theory. The great mathematician David Hilbert was prophetic when he declared, "*No one shall expel us from the Paradise that Cantor has created.*"

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Before Cantor arrived on the scene, progress in mathematics was often being achieved at the expense of rigour. Sometimes mathematicians were unable to formulate precise definitions and appealed to intuition or geometric pictures. For instance, the only treatment of real numbers was the geometric one as 'points on a line'. Apart from the compromise on rigour, there were other limitations. For instance, the only functions that were being considered were those which had analytic expressions. Cantor's work led to the foundation of mathematics and revived the ancient Greek ideas of rigour and precision in mathematics. It also led to fruitful generalisations and opened up new possibilities in mathematics.

Keywords
Trigonometric series, sets of uniqueness, accumulation points, derived sets, power of a set, ordinal numbers, transfinite induction.

It is interesting to note that Cantor was led to this discovery by attacking a precise problem in the then emerging area of trigonometric series initiated by Joseph Fourier at the dawn of the nineteenth century.

For the convenience of readers, we will not present the discoveries made by Cantor in their chronological order. We will consider several problems that Cantor considered and describe only his final results. We will present them in modern language and not in the language that Cantor had used.

2. Fourier Series

First we state a few results from analysis that give the background in which Cantor started working.

For a function $f : (a, b) \to \mathbb{R}$, we define

$$\Delta^2 f(x, h) = f(x + h) + f(x - h) - 2f(x).$$

Theorem 1 (Schwarz). *Suppose f is continuous and*

$$D^2 f(x) = \lim_{h \to 0} \frac{\Delta^2 f(x, h)}{h^2}$$

exists for all x in an open interval (a, b). If $D^2 f \geq 0$ on (a, b), then f is convex. In particular, if $D^2 f = 0$ on (a, b), f is linear on (a, b).

Note that if f is twice continuously differentiable, $D^2 f = f''$, the second derivative of f. Thus, the above result is a generalization of a standard result on twice differentiable functions.

Now consider a formal trigonometric series (also known as Fourier series)

$$S \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad x \in \mathbb{R}. \qquad (*)$$

We say that the series is convergent for a real number x

if

$$\lim_{N\to\infty}\sum_{n=-N}^{N} c_n e^{inx}$$

exists.

The convergence of a Fourier series is difficult to determine. The German mathematician Bernhard Riemann proved two important results to circumvent this problem. We can formally integrate the series termwise twice and get a tractable function. So, consider

$$f(x) = \frac{c_0}{2}x^2 - \sum_{n\neq 0}\frac{c_n}{n^2}e^{inx}, \quad x \in \mathbb{R}.$$

Since $\sum_{n>0}\frac{1}{n^2} < \infty$, we easily see that by the well-known Weierstrass M-test, the infinite series in the expression for $f(x)$ is uniformly convergent, provided the sequence $\{c_n\}$ is bounded. Hence, if $\{c_n\}$ is bounded, f is a continuous function on the set of real numbers $\mathbb{R}$.

Here are the two results of Riemann.

Theorem 2 (Riemann's First Lemma). *If the coefficients $\{c_n\}$ of the Fourier series S are bounded and $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges to s for some $x \in \mathbb{R}$, then $D^2 f(x)$ exists and equals s.*

Theorem 3 (Riemann's Second Lemma). *If $c_n \to 0$ as $n \to \infty$, then*

$$\lim_{h\to 0}\frac{\Delta^2 f(x,h)}{h} = 0$$

for every $x \in \mathbb{R}$.

We have now given the relevant background when Cantor arrived on the scene. The following problem was suggested to Cantor by one of his colleagues Eduard Heine:

It was Cantor who answered the question of unique representation of a function by trigonometric series in the affirmative in complete generality, a remarkable landmark of that time.

"Suppose the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges to 0 pointwise for all real numbers x. Must then each a_n and each b_n be equal to 0?"

Setting

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2} \text{ and } \sin nx = \frac{e^{inx} - e^{-inx}}{2i},$$

we can replace the series by a series of the form

$$S \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad x \in \mathbb{R}. \qquad (*)$$

The question is: If S converges to 0 for every real number x, must each c_n be equal to 0?

This problem was attempted by the great minds of that time including Peter Dirichlet, Eduard Heine, Rudolph Lipschitz and Bernhard Riemann. But they could give an affirmative answer only for some specific cases and that too under certain restrictions. It was Cantor who answered the above question in the affirmative in complete generality, a remarkable landmark of that time. Thus, Cantor proved the following result.

Theorem 4 (1870). *Every function $f : \mathbb{R} \to \mathbb{R}$ can have at most one representation by a trigonometric series.*

Mathematicians of his time were attacking the uniqueness problem in this form. They obtained results only for very special classes of functions and only when the convergence was uniform. For instance, the best result till then, due to Heine, was for all continuous functions f under the additional hypothesis of uniform convergence of the trigonometric series.

Here is the proof of Cantor. He first proved this lemma.

Lemma 5. *If S converges to* 0 *for all real numbers x, then $c_n \to 0$.*

Now formally integrate the Fourier series S termwise twice to get $f(x)$ as above. By Riemann's first lemma, $D^2 f(x) = 0$ for all real x. By Schwarz's theorem, f is linear. Let

$$f(x) = \frac{c_0}{2}x^2 - \sum_{n \neq 0} \frac{c_n}{n^2} e^{inx} = m \cdot x + l, \quad x \in \mathbb{R}.$$

By substituting $x = \pi$ and $-\pi$, and subtracting, we get $m = 0$. Next substitute $x = 0$ and $2 \cdot \pi$ and subtract to prove that $c_0 = 0$. So, now we have the identity

$$\sum_{n \neq 0} \frac{c_n}{n^2} e^{inx} = -l.$$

Since the series is uniformly convergent, interchanging the order of integration and summation is possible. Thus, for each non-zero integer k, we get

$$2\pi \frac{c_k}{k^2} = \sum_{n \neq 0} \int_{x=0}^{2\pi} \frac{c_n}{n^2} e^{i(n-k)x} \mathrm{d}x = -\int_{x=0}^{2\pi} l \cdot e^{-ikx} \mathrm{d}x = 0,$$

implying that $c_k = 0$.

3. Sets of Uniqueness of Fourier Series

Cantor did not stop with Theorem 4. Using Riemann's second lemma, he could see that the hypothesis of the convergence of the trigonometric series can be relaxed on some exceptional sets of real numbers. We first give a definition before stating the first such result. This led Cantor to introduce some general topological notions and develop the theory of ordinal numbers.

DEFINITION.

Call a set D of real numbers *a set of uniqueness* if, whenever the trigonometric series in $(*)$ converges to 0 pointwise for all real numbers $x \in \mathbb{R} \setminus D$, each c_n equals 0.

Cantor's theorem says that the empty set is a set of uniqueness. Note that a subset of a set of uniqueness is a set of uniqueness.

Cantor observed that Lemma 5 is true even when S converges to 0 for all x in a non-empty open interval.

Lemma 6. If S converges to 0 for all x in a non-empty open interval, then $c_n \to 0$ as $n \to \infty$.

Using Riemann's second lemma, Cantor generalized Theorem 4 as follows:

Theorem 7 (1871). *Any collection of real numbers of the form*

$$\cdots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \cdots$$

such that $\inf x_n = -\infty$ *and* $\sup x_n = \infty$ *is a set of uniqueness.*

Proof. By the above argument, for each integer i, f is linear on (x_i, x_{i+1}). Say,

$$f(x) = a \cdot x + b, \quad x_{i-1} < x < x_i,$$

and

$$f(x) = c \cdot x + d, \quad x_i < x < x_{i+1}.$$

By Riemann's second lemma,

$$\lim_{h \to 0} \frac{f(x_i + h) + f(x_i - h) - 2f(x_i)}{h} = 0,$$

or

$$\lim_{h \to 0+} \frac{f(x_i + h) - f(x_i)}{h} - \lim_{h \to 0+} \frac{f(x_i - h) - f(x_i)}{-h}$$

$$= c - a = 0.$$

Thus, the slopes of f on each segment (x_i, x_{i+1}) are the same. Since f is continuous, it follows that f is linear on $\mathbb{R}$. Hence, as proved in Theorem 4, each $c_n = 0$.

4. Accumulation Points and Derived Set

Although Cantor used an exceptional set for the first time in the above theorem, he was till then essentially working within the prevalent traditions of mathematics. He used only the tools and insights developed by his contemporaries such as Heine, Riemann, Schwarz and Weierstrass. Cantor now made further investigations on exceptional sets and came up with innovative constructions. His seminal ideas ultimately led to the creation of set theory.

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Influenced by the notion of 'condensation point' (also known as *limit or accumulation point* today) of a set of real numbers introduced by Heine, Cantor introduced the notion of the derived set of a set of real numbers:

For $P \subset \mathbb{R}$, Cantor defined the 'derived set' of P by

$$P' = \{x \in \mathbb{R} : x \text{ is an accumulation of } P\}.$$

For sets D in Theorem 7, $D' = \emptyset$. Cantor improved Theorem 7 as follows.

Theorem 8. *If $P'' = \emptyset$, P is a set of uniqueness.*

Proof. Since $P'' = \emptyset$, P' is a subset of a set

$$\{\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots\}$$

satisfying the hypothesis of Theorem 7. Fix an integer i and take a compact interval $[m, M] \subset (x_i, x_{i+1})$. Since $[m, M]$ is compact and contains no accumulation point of P, by the Bolzano–Weierstrass theorem, the compact interval $[m, M]$ will contain at most finitely many points from P, say

$$m \leq a_1 < a_2 < \cdots a_k \leq M.$$

Now, by the above argument, f is linear on each (a_i, a_{i+1}), with all slopes equal by Riemann's second lemma. So, f

is linear on $[m, M]$. This implies that f is linear on each (x_i, x_{i+1}). Applying Riemann's second lemma again, we see that all slopes are equal. Hence, f is linear on $\mathbb{R}$. The result now follows as before.

Indeed, Cantor proved that his arguments can be easily extended to more general sets. For any set P of real numbers, Cantor inductively defined

$$P^{(0)} = P, P^{(n+1)} = (P^{(n)})',$$

n a natural number. He called a set P of real numbers, *a set of first species* if for some natural number n, $P^{(n)} = \emptyset$. Cantor could see the following result.

Theorem 9. *Every set of first species is a set of uniqueness.*

The proof of Theorem 9 required a fundamentally different way of looking at real numbers. Thus, Cantor set out to develop a satisfactory theory of real numbers.

The proof of this theorem required a fundamentally different way of looking at real numbers. In the following section, we discuss how Cantor viewed the real numbers and proved the above theorem.

5. Cantor's Definition of Real Numbers

As mentioned earlier, Cantor faced difficulty in presenting his results precisely with the mathematical language developed up to that time. Till then, real numbers were informally understood as points on a line. It was difficult to define the sets of first species precisely with this informal picture alone. Cantor soon realised that he needed to define real numbers, so to speak, arithmetically and not merely as points on a line. Thus, Cantor set out to develop a satisfactory theory of real numbers.

Like his contemporaries Dedekind and Weierstrass, Cantor accepted that rational numbers are precisely defined through arithmetic. Let $\mathbb{Q}$ denote the set of all rational numbers. It has a natural ordering and arithmetical operations (making it into an ordered field). The notion of distance between two rational numbers is also precisely understood.

In order to define a real number arithmetically, Cantor now considered a sequence $\{r_n\}$ of rational numbers satisfying the property: given a rational number $\epsilon > 0$, there is a natural number N such that for every pair of natural numbers $n, m > N$, $|r_n - r_m| < \epsilon$. Such a sequence was called a *fundamental sequence* by Cantor; it is now called a 'Cauchy sequence'.

Cantor identified two such sequences $\{r_n\}$ and $\{s_n\}$ if $|r_n - s_n| \to 0$ as $n \to \infty$. We now say that two Cauchy sequences $\{r_n\}$ and $\{s_n\}$ are equivalent if $|r_n - s_n| \to 0$ as $n \to \infty$. Let $[\{r_n\}]$ be the set of all Cauchy sequences $\{s_n\}$ of rational numbers equivalent to $\{r_n\}$. Cantor defined a *real number* to be the equivalence class of a Cauchy sequence of rational numbers. A rational number r was identified with the equivalence class containing the constant sequence $\{r, r, r, \ldots\}$.

It was easy to extend the ordering, arithmetical operations and the distance function on $\mathbb{Q}$ to the set of real numbers $\mathbb{R}$ thus defined. Let x and y be two real numbers represented by Cauchy sequences of rational numbers $\{r_n\}$ and $\{s_n\}$ respectively. It was shown that $\{|r_n - s_n|\}$ is a Cauchy sequence of rational numbers. The real number represented by it was called the *distance* between x and y; it was denoted by $|x - y|$. He also defined

$$x + y = [\{r_n + s_n\}]$$

and

$$x \cdot y = [\{r_n \cdot s_n\}].$$

Cantor observed the following trichotomy property – exactly one of the following three properties holds:

- There is a negative rational number r such that $r_n - s_n < r$ for all large n.

- $r_n - s_n \to 0$ as $n \to \infty$.

- There is a positive rational number r such that $r_n - s_n > r$ for all large n.

Thinking of a real number as an equivalence class of a Cauchy sequence of real numbers was a very bold step and a profound contribution to mathematics.

Cantor defined $x < y$ if (1) is satisfied. These are easily seen to be well-defined.

Thinking of a real number as an equivalence class of a Cauchy sequence of rational numbers was a very bold step and a profound contribution to mathematics. It had a significant impact on the theory of functions and other areas.

In order to describe accumulation points of a set of real numbers, Cantor initially viewed them as the equivalence classes of Cauchy sequences of real numbers from the set. The process was iterated to define higher order derived sets of the set. But Dedekind pointed out to Cantor that equivalence classes of real numbers led only to real numbers.

More precisely, it was shown that for every Cauchy sequence of real numbers $\{x_n\}$, there is a Cauchy sequence $\{r_n\}$ of rational numbers such that $|x_n - r_n| \to 0$ as $n \to \infty$. In present day terminology, it means that the set of real numbers is 'complete'. In fact, this is how the *completion of a metric space* is defined today.

After defining real numbers and distances between them, Cantor made topological notions of accumulation points, derived sets, dense sets, nowhere dense sets etc. precise. Cantor also introduced the notion of *everywhere dense* and *nowhere dense* sets. For instance, in a set P of real numbers, P is called nowhere dense, if every non-empty open interval contains an open, proper subinterval that does not contain any point of P. Cantor noted that all these concepts made sense in every Euclidean space $\mathbb{R}^n$.

With these developments, sets of first species were precisely defined. We now present Cantor's proof of Theorem 9.

Let P be a closed set of real numbers of first species.

Suppose

$$S \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

converges to 0 pointwise for every $x \in \mathbb{R} \setminus P$. As before, set

$$f(x) = \frac{c_0}{2}x^2 - \sum_{n \neq 0} \frac{c_n}{n^2} e^{inx} = 0.$$

We need to show that f is linear on $\mathbb{R}$. For each natural number n, set

$$V^n = \mathbb{R} \setminus P^{(n)}.$$

By induction on n, we show that f is linear on each open interval contained in V^n. Since $P^{(n)} = \emptyset$ for some n, $V^n = \mathbb{R}$ for that n. This will complete the proof.

Since S vanishes outside P, as shown earlier, f is linear on every open interval contained in V^0. Now suppose f is linear on every open interval contained in V^n. Let $[m, M]$ be a compact interval contained in V^{n+1}. By the Bolzano–Weierstrass theorem, $[m, M]$ contains at most finitely many points, say

$$m \leq a_1 < \cdots < a_k \leq M,$$

of $P^{(n)}$. Then each (a_i, a_{i+1}) is contained in V^n, Hence, by induction hypothesis, f is linear on each (a_i, a_{i+1}). Using Riemann's second lemma, we show that f is linear on $[m, M]$.

6. The Concept of Denumerability

By now, Cantor had achieved spectacular success with the uniqueness problem of trigonometric series. After this achievement, Cantor's interest shifted from trigonometric series to development of the theory of real numbers, which he had introduced. To express a property on the sizes of the sets of first species, he now introduced a radically new concept: he called a set *denumerable* if its elements could be arranged as a sequence $\{x_n\}$. Cantor proved the following result.

Theorem 10. *The set of isolated points of a set of real numbers is denumerable.*

We shall see that the introduction of the notion of denumerability marked the beginning of a profound creation in mathematics. Cantor proved several results on denumerable sets which are now called *countable sets*. We list a few.

Theorem 11.

1. *The set of ordered pairs of natural numbers* $\mathbb{N} \times \mathbb{N}$ *is countable.*
2. *A subset of a countable set is countable.*
3. *If* A *and* B *are countable, so is their cartesian product* $A \times B$*.*
4. *If* $A_1, A_2, \cdots$ *are countable sets, so is their union* $\cup_n A_n$*.*

These observations immediately implied the following result.

Theorem 12. *The set of all rational numbers, the set of all algebraic numbers, and all sets of first species are countable.*

Proof. We prove only the last part of the result. Let P be a closed set of real numbers such that there exists a positive integer n with $P^{(n)} = \emptyset$. Then

$$P = \cup_{i<n}(P^{(i)} \setminus P^{(i+1)}).$$

Now note that each $P^{(i)} \setminus P^{(i+1)}$ is countable by Theorem 10.

Starting with these results, Cantor gradually began to develop set theory abstractly. He next asked, "Is the set of all real numbers countable?" He discovered that it is not and gave a brilliant proof of it. He was extremely excited with the discovery. It revealed to him that there were different kinds of infinities, i.e., infinities with different 'magnitudes'. We recall his proof below. We call an interval non-degenerate if it has more than one points.

Theorem 13. *Every non-degenerate interval I is uncountable.*

Proof. Suppose I is countable. Fix an enumeration $x_0, x_1, x_2, \cdots$ of I. Without any loss of generality, we assume that $x_0 < x_1$. Set $a_0 = x_0$ and $b_0 = x_1$.

There must be an element $x \in I$ such that $a_0 < x < b_0$. Let a_1 be the first element in the enumeration such that $a_0 < a_1 < b_0$. Next choose the first element b_1 in the enumeration such that $a_1 < b_1 < b_0$. We proceed similarly and get

$$a_0 < a_1 < a_2 < \cdots\cdots < b_2 < b_1 < b_0$$

such that for each n, a_{n+1} is the first element in the enumeration such that $a_n < a_{n+1} < b_n$ and b_{n+1} is the first element in the enumeration such that $a_{n+1} < b_{n+1} < b_n$. Let $x = \sup a_n$. Clearly x belongs to I and is different from each x_i, a contradiction. Hence the result.

An Interesting Remark. It is interesting to note that the title of the paper where Cantor proved uncountability of the set of all real numbers was 'On a property of the collection of all real algebraic numbers.' Some traditional mathematicians like Kronecker were highly critical of the abstract approach of Cantor and his use of infinite sets. To hide his very abstract approach, Cantor chose such a title.

Cantor's deductions from proving Theorem 13 was the first proof of the existence of a transcendental number.

From this theorem Cantor deduced that each non-degenerate interval contains uncountably many transcendental numbers. *This was the first proof of the existence of a transcendental number.*

7. Cantor and Set Theory

We saw that starting with trigonometric series, Cantor very naturally started thinking of arbitrary sets quite abstractly. Next, Cantor took up the problem of comparing the sizes of two infinite sets and to formulate it in a precise mathematical language. He defined two sets X and Y to have *the same power* if there is a one-to-one correspondence between their elements. Cantor developed this concept as we know it today. We use the notation $|X| = |Y|$ if the sets X and Y have the same power, and $|X| \leq |Y|$ if there is a one-to-one map from X into Y. Further, we write $|X| < |Y|$ if $|X| \leq |Y|$ and $|X| \neq |Y|$. Following are some of the prominent results that Cantor proved.

Theorem 14.

1. *For every set X, $|X| < |\mathcal{P}(X)|$, where $\mathcal{P}(X)$ denotes the power set of X.*

2. *For any two sets X and Y, either $|X| \leq |Y|$ or $|Y| \leq |X|$.*

3. *For any infinite set X, $|X \times X| = |X|$.*

Cantor was very excited with the last conclusion. This implied that all Euclidean spaces $\mathbb{R}^n$ of different dimensions have the same power. In astonishment, he wrote, "I see it but don't believe it." He thought that he had solved the dimension problem of topology. But Richard Dedekind pointed out to him that his proof did not demonstrate the existence of any continuous bijection between Euclidean spaces of different dimensions. We now know that there does not exist a continuous one-to-one function from $\mathbb{R}^n$ to $\mathbb{R}^m$ if $m < n$.

Theorem 14 implied that all Euclidean spaces **R**^{n} of different dimensions have the same power. In astonishment, Cantor wrote, "*see it but don't believe it*."

Cantor used the method of transfinite induction (which we shall describe later) to prove these results. Cantor also conjectured but could not prove the following result.

Theorem 15. For any two sets X and Y, if $|X| \leq |Y|$ and if $|Y| \leq |X|$, then $|X| = |Y|$.

This theorem is now known as the Cantor–Schröder–Bernstein theorem. A proof of the result was communicated to Cantor by Dedekind. However, Dedekind's letter was lost for sometime. Bernstein and Schröder independently gave a proof of this result before Dedekind's letter resurfaced.

Cantor continued his study of exceptional sets such as sets of the first species that he had introduced to study the uniqueness problem of trigonometric series. Cantor gave examples of sets P such that for no natural number n, $P^{(n)} = \emptyset$. Of course, the set of all real numbers and the set of all rational numbers are such sets. But Cantor had something more in his mind. For any set P of real numbers, he defined

$$P^{(\infty)} = \cap_n P^{(n)},$$

$$P^{(\infty+1)} = (P^{(\infty)})',$$

$$P^{(\infty+2)} = (P^{(\infty+1)})',$$

and so on. He could give examples of sets P such that $P^{(\infty)} \neq \emptyset$ but $P^{(\infty+1)} = \emptyset$, or $P^{(\infty+1)} \neq \emptyset$ but $P^{(\infty+2)} = \emptyset$, and so on. Cantor saw that this process could be continued indefinitely. He defined

$$P^{(\infty+\infty)} = \cap_n P^{(\infty+n)},$$

$$P^{(\infty+\infty+1)} = (P^{(\infty+\infty)})',$$

$$P^{(\infty+\infty+2)} = (P^{(\infty+\infty+1)})',$$

and so on. Cantor called a set P of real numbers to be a *set of second species* if all sets obtained as above are non-empty. But all these concepts had to be made precise.

Cantor called a non-empty set of real numbers *dense-in-itself*, if it has no isolated point and *perfect*, if it is dense-in-itself and if every Cauchy sequence in it converges to

In order to make the process of indefinitely going on taking derivatives of a set precise, Cantor was led to develop the theory of transfinite numbers.

a point in it. Around 1887, Paul du Bois–Reymond and Axel Hernack had introduced the notion of a set of real numbers to be of *content* (or *measure*) *zero* in their study of the theory of integrals. A set $P \subset \mathbb{R}$ is of content zero, if for every $\epsilon > 0$, there is a sequence of open intervals $\{(a_n, b_n)\}$ covering P such that $\sum_n (b_n - a_n) < \epsilon$. Cantor introduced his famous *Cantor ternary set* to show that there are nowhere dense sets of second species and that there are sets of second species of measure zero. He also gave examples of perfect, nowhere dense sets of positive content.

To develop the theory of ordinal numbers, Cantor introduced well-ordered sets. A *well-ordered set* is a linearly ordered set $(W, \leq)$ such that every non-empty tail subset (a set $T \subset W$ such that whenever $x \in T$ and $x \leq y$, $y \in T$) has a first element. This means that after any set of stages, there is a next or a first stage. The set of all natural numbers is well-ordered, since, after every natural number, there is a next natural number. This cannot be said of the set of all rational numbers. By introducing well-ordered sets, Cantor was able to capture the first stage after every finite stage, the stage next to it, and so on.

Cantor identified all isomorphic well-ordered sets as one and called them *ordinal numbers*. If $(W, \leq)$ is a well-ordered set and $w \in W$, the set $W(w) = \{v \in W : v < w\}$ is called an *initial segment* of W.

8. Ordinals and Transfinite Induction

Cantor extended the method of induction to induction on arbitrary well-ordered sets. This was yet another fundamental contribution of Cantor to mathematics.

Cantor first extended the method of induction to induction on arbitrary well-ordered sets. This was yet another fundamental contribution of Cantor to mathematics. For natural numbers, one had the following methods of induction.

Theorem 16. *Suppose that for every natural number n, A_n is a statement such that A_0 is true and whenever*

A_n is true, A_{n+1} is true. Then, for every n, A_n is true.

Theorem 17. *Suppose that X is an arbitrary set, $x_0 \in X$ and $G : X \to X$ a function. Then there is a unique function $f : \mathbb{N} \to X$ such that*

$$f(0) = x_0 \ \& \ f(n+1) = G(f(n)), \quad n \in \mathbb{N}.$$

Note that in a well-ordered set, there may be an element that does not have an immediate predecessor. For instance, consider $W = \{1 - \frac{1}{n} : n \geq 1\} \cup \{1\}$ with the usual order. This is a well-ordered set such that 1 has no immediate predecessor. For this reason we have the following:

Theorem 18. *Let $(W, \leq)$ be a well-ordered set and for each $w \in W$, A_w be a statement. Assume that for every $w \in W$, A_w is true, whenever A_v is true for all $v < w$. Then, for every $w \in W$, A_w is true.*

Proof. Set

$$T = \{w \in W : A_w \text{ is not true}\}.$$

If possible, suppose $T \neq \emptyset$. Let w be the least element of T. But then for every $v < w$, A_v is true. Hence, by our hypothesis, A_w is true. This contradicts that $w \in T$ and our result is proved.

Similarly, Theorem 17 for arbitrary well-ordered sets is suitably formulated and proved. Let $\mathcal{F}$ denote the set of all functions with domain an initial segment of W and range contained in X. Take a function $G : \mathcal{F} \to X$. We may think of G as a 'rule' that gives the value of a function f at a point $w \in W$ if we know $f|W(w)$, the restriction of f to the initial segment $W(w)$.

Theorem 19. *For every $G : \mathcal{F} \to X$, there is a unique function $f : W \to X$ such that for every $v \in W$*

$$f(v) = G(f|W(v)). \qquad (**)$$

Proof. For each $w \in W$, let A_w be the statement, "There is a unique function $f_w : W(w) \to X$ such that $(**)$ is satisfied by $f = f_w$ for every $v \in W(w)$." By transfinite induction on W, one easily proves that for each $w \in W$, A_w is true. By the uniqueness, one sees that f_w's are consistent. Hence, we get a function $f : \cup_{w \in W} W(w) \to X$ that extends each f_w. If W has no greatest element, $W = \cup_{w \in W} W(w)$ and we are done. Otherwise, suppose w' is the greatest element of W. Then $W(w') = \cup_{w \in W} W(w)$. Now define $f(w') = G(f)$.

Cantor observed:

Theorem 20. *W is not order isomorphic to any of its initial segments.*

Proof. Suppose there exists a $w \in W$ and an order preserving injection $f : W \to W(w)$. Set $w_0 = w$ and, for every natural number n, $w_{n+1} = f(w_n)$. Then $w_0 > w_1 > w_2 > \ldots$. Hence, the set $\{w_n\}$ is non-empty with no least element. This is a contradiction. Thus, our result is proved.

Using transfinite induction, Cantor proved the following trichotomy theorem:

Theorem 21. *Let $(W, \leq)$ and $(W', \leq')$ be two well-ordered sets. Then, exactly one of the following three conditions holds:*

- *$(W, \leq)$ is isomorphic to an initial segment of $(W', \leq')$.*
- *$(W, \leq)$ is isomorphic to $(W', \leq')$.*
- *$(W', \leq')$ is isomorphic to an initial segment of $(W, \leq)$.*

Let α and β be two ordinal numbers represented by well-ordered sets $(W, \leq)$ and $(W', \leq')$ respectively. We set $\alpha < \beta$ if W is order isomorphic to an initial segment of W'. This defines a linear ordering on the class of all ordinal numbers. It was also shown that:

Theorem 22. *Every non-empty set of ordinal numbers contains a least ordinal number.*

This means that every set of ordinal numbers is a well-ordered set.

Cantor noted that for each natural number n, any two well-ordered sets with n elements are order-isomorphic. So, for each n, there is exactly one ordinal number with n elements. This was denoted by n itself.

Next he showed that the set of all natural numbers with the usual ordering can be treated as the first infinite ordinal number and denoted it by ω. Thus, he removed the use of the obscure symbol ∞ that he used initially.

For any ordinal α represented by a well-ordered set, say $(W, \leq)$, Cantor added a new element, say y, to W and extended $\leq$ to $W \cup \{y\}$ by defining y larger than every element $x \in W$. He denoted the corresponding ordinal number by $\alpha + 1$. Ordinal numbers obtainable in this manner were called *successor ordinals*. Other ordinal numbers were called *limit ordinals*.

Cantor could also extend the arithmetic of natural numbers to the arithmetic of ordinal numbers. But we shall only define the sum of two ordinal numbers α and β. Let α and β be represented by $(W', \leq')$ and $(W'', \leq'')$ respectively. Without any loss of generality, we assume that W' and W'' are disjoint. Let $W = W' \cup W''$. For $u, v \in W$, set $u \leq v$ if any one of the following three conditions is satisfied:

$$(u, v \in W' \ \&\ u \leq' v), (u, v \in W'' \ \&\ u \leq'' v)\ ,$$

$$(u \in W' \ \&\ v \in W'')\ .$$

It is easy to check that $(W, \leq)$ is a well-ordered set. The corresponding ordinal number is denoted by $\alpha + \beta$. With this notation, it is easy to see that:

Theorem 23.

$$1 + \omega = \omega < \omega + 1.$$

The arithmetic of ordinal numbers restricted to finite ordinals coincided with the usual arithmetic of natural numbers. That is why Cantor also called ordinal numbers *transfinite numbers.*

Cantor showed that the process of taking successive derivatives of a set of real numbers must stabilize at a countable stage. He called an ordinal number α *a countable ordinal* if the underlying set of a well-ordered set representing α is countable. Let Ω denote the set of all countable ordinals.

Using transfinite induction, Cantor precisely defined successive derivatives of a set P of real numbers as follows:

$$P^{(0)} = P,$$

$$P^{(\alpha+1)} = (P^{(\alpha)})'$$

and for limit ordinals λ,

$$P^{(\lambda)} = \cap_{\alpha<\lambda} P^{(\lambda)}.$$

Cantor proved the following result.

Theorem 24. *For every set P of real numbers, there is a countable ordinal α_0 such that for every $\beta > \alpha_0$, $P^{(\beta)} = P^{(\alpha_0)}$.*

Proof. Let $\{(r_n, s_n)\}$ be an enumeration of all bounded, open intervals with rational end points. For each α such that $P^{(\alpha)} \setminus P^{(\alpha+1)} \neq \emptyset$, choose an integer $n = n(\alpha)$ such that $(r_n, s_n) \cap P^{(\alpha)} \neq \emptyset$ and $(r_n, s_n) \cap P^{(\alpha+1)} = \emptyset$. This defines a one-to-one map from $A = \{\alpha \in \Omega : P^{(\alpha)} \setminus P^{(\alpha+1)} \neq \emptyset\}$ into $\mathbb{N}$. Thus, A is countable. Now take $\alpha_0 \in \Omega$ to be the first ordinal bigger than all $\alpha \in A$.

Sets for which $P^{(\alpha_0)} = \emptyset$ were called *sets of first species*; others were called *sets of second species*. It is easy to check that if P is a closed set of real numbers, then

$$P = \cup_{\alpha<\alpha_0}(P^{(\alpha)} \setminus P^{(\alpha+1)}) \cup P^{(\alpha_0)}.$$

In particular, if $P^{(\alpha_0)} = \emptyset$, $P = \cup_{\alpha<\alpha_0}(P^{(\alpha)} \setminus P^{(\alpha+1)})$. By Theorem 10, each $P^{(\alpha)} \setminus P^{(\alpha+1)}$ is countable. This implies that sets of first species are countable.

Further, if $P^{\alpha_0} \neq \emptyset$, it is non-empty, closed and dense-in-itself. Any such set of real numbers is uncountable. Thus, we have the following result.

PROPOSITION 25.

Let $P \subset \mathbb{R}$ be a closed set. Then P is countable, if and only if, for some countable ordinal α_0, $P^{(\alpha_0)} = \emptyset$.

Although Cantor's interest in these sets was more set-theoretic and topological, it is interesting to note the following result.

Theorem 26. *Countable closed sets and the Cantor ternary set are sets of uniqueness.*

It is quite likely that Cantor had seen that every countable closed set is a set of uniqueness. We sketch a proof of this now.

Let P be a countable closed set of real numbers. Let α_0 be the first countable ordinal such that $P^{(\alpha_0)} = \emptyset$. For $\alpha \leq \alpha_0$, set

$$V^\alpha = \mathbb{R} \setminus P^{(\alpha)}.$$

By transfinite induction, we show that f is linear on each interval contained in V^α. Since $V^{\alpha_0} = \mathbb{R}$, this will complete the proof.

For $\alpha = 0$, since S vanishes outside P, S vanishes on each interval contained in $\mathbb{R} \setminus P$. So, the hypothesis for $\alpha = 0$ follows by an earlier argument.

Assuming that f is linear on each component of V^{α}, using the Bolzano–Weierstrass theorem as before, we show that f is linear on each compact interval $[m, M]$ contained in $V^{\alpha+1}$.

Now, assume that $\lambda \leq \alpha_0$ is a limit ordinal and for every $\alpha < \lambda$, f is linear on all intervals contained in V^{α}. We have

$$V^{\lambda} = \cup_{\alpha<\lambda} V^{\alpha}.$$

Now, note that each compact interval $[m, M]$ contained in V^{λ} is contained in V^{α} for some $\alpha < \lambda$. Hence, by induction hypothesis, f is linear on $[m, M]$. This completes the proof.

It has been shown that every countable set of real numbers, not necessarily closed, is a set of uniqueness. It is now proved that every set of uniqueness is a set of measure zero. Rademacher has shown that the Cantor ternary set is a set of uniqueness. It is easy to construct a perfect, nowhere dense set of real numbers of positive measure. So, not all perfect, nowhere dense set is a set of uniqueness.

Even before Cantor arrived on the scene, there was a resistance to the idea of completed infinity in mathematics. For instance, even Gauss had announced, "The notion of completed infinity does not belong in mathematics."

Remark. So far we have given only the success story of Cantor. Even before Cantor arrived on the scene, there was a resistance to the idea of completed infinity in mathematics. For instance, even Gauss had announced, "The notion of a completed infinity does not belong in mathematics." To add to Cantor's miseries, his mathematics was not well-received by some prominent mathematicians like Kronecker and Poincaré. For instance, Kronecker had said, "I don't know what predominates in Cantor's theory – philosophy or theology, but I am sure that there is no mathematics there."

Cantor was very frustrated by his inability to settle continuum hypothesis and well-ordering principle.

Apart from these philosophical criticisms, Cantor met with some notable failures too. He was very frustrated by his inability to settle the continuum hypothesis and the well-ordering principle. More specifically, he tried very hard to prove that every uncountable set of real numbers has the same power as that of $\mathbb{R}$. He did not formulate the well-ordering principle. But he attempted and failed to give a well-ordering of $\mathbb{R}$. Also, Cantor's naive approach led to some paradoxes such as Russell's paradox, Richard's paradox, etc. He seemed to be quite disturbed by them and did not know how to settle them.

Unfortunately, Cantor did not live long enough to see the surprising solutions of these problems. It turned out that they are related to the foundations of mathematics itself. This forced the development of set theory axiomatically and not naively, as Cantor had done. Finally, using very intricate and powerful techniques, Kurt Gödel and Paul Cohen showed that the well-ordering principle and the continuum hypothesis are independent of the axioms of set theory.

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Suggested Reading

For an introduction to cardinal numbers, one may see

[1] S M Srivastava, Transfinite numbers. what is infinity, *Resonance*, Vol.2, No.3, 1997.

For a quick introduction to both cardinal and ordinal numbers, one may see the first chapter of:

[2] S M Srivastava, A Course on Borel Sets}, GTM 180, Springer, New York, 1994.

For historical comments, see

[3] J W Dauben, *Georg Cantor, His Mathematics and Philosophy of the Infinite*, Princeton University Press, 1990.

For more on the role of set theory in the study of the structures of sets of uniqueness, see

[4] A S Kechris and Alain Louveau, Descriptive Set Theory and the Structure of Sets of Uniqueness, London Mathematical Society Lecture Note Series 128, Cambridge University Press, 1987.

For an introduction to set theory, we recommend

[5] H B Enderton, Elements of set theory, Academic Press, 1977.

For axiomatic set theory, we suggest

[6] K Kunen, Set Theory, An Introduction to Independence Proofs, Studies in Logic and the Foundations of Mathematics, 102, North-Holland Publishing Company, 1980.

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