Crossing a Nonlinear Resonance
Adiabatic Invariants and the Melnikov–Arnold Integral

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The idea of adiabatic invariance is presented in the context of simple classical mechanical models. The adiabatic invariant jumps across the separatrix – an attempt has been made to bring out the basic ideas underlying the Melnikov–Arnold integral. This becomes important as soon as a perturbation to a regular, stable system makes it dynamically unstable.

1. Introduction to the Mechanics in One Degree of Freedom

Consider the planetary system. A lone planet under the influence of Sun will move along an elliptic orbit with some velocity. Other planets perturb these orbits, giving the perturbed velocity $\frac{dx}{dt} = v(x) + \epsilon v_1(x)$, where $v(x)$ is the unperturbed velocity. The role of $\epsilon$ is played by the ratio of masses of planets to that of the Sun (roughly, of order $1/1000$). Time of order $1/\epsilon$ is already a time-span of thousands or tens of thousands of years. Thus, even slowly, the slow motion could double the extent of Earth’s orbit over such a long time-period. This would be fatal for our civilization. The underlying classic problem has held the attention of mathematicians like Lagrange, Laplace, and Gauss. Quoting Arnold [1], Gauss formulated the problem of planetary motion in the following way: “... to determine evolution, one has to smear the mass of each planet over the orbit in proportion to the time spent in every part of the orbit and replace the attraction of planets by the attraction of rings thus obtained.”

The two-body problem between gravitating bodies has a stable dynamics and thus an eternal existence. However,
Adiabatic invariants are very important quantities, relevant to fields ranging from classical to quantum physics.

with an additional body, the dynamics can be very complicated. Does it become completely unstable? Not really, otherwise we wouldn’t be here. The planetary motion exhibits features of instability and complexity but the perturbations are just about ‘tolerable’ – allowing the existence for long enough times. Thus, for instance, the motion of earth is not a perfect ellipse, as it experiences the gravitational pull of other planets also. However, if we allow the mass of the bodies moving about the Sun to become smaller, and we get to the asteroid masses, these objects may be influenced enough that the perturbation is more pronounced. Indeed, the belt of asteroids between Mars and Jupiter arranges itself in belts with specific gaps originating from the perturbation caused by Jupiter’s pull as they revolve around the Sun.

The physical situations where there is more than one timescale – slow and fast – are common. In addition to the one discussed above, it also appears in plasma physics, astrophysics, accelerator physics, nuclear and condensed matter physics, and so on. For instance, Fermi’s theory for the acceleration of cosmic rays [2] assumes that the magnetic moment of a particle spiralling about a magnetic field line is almost constant. The magnetic moment does not change much over a Larmor radius, and is what is called an adiabatic invariant [3]. The adiabatic invariant for a simple problem, a particle being reflected from a slowly moving wall, is explained in Box 1. In the quantum mechanical context, the distribution of energy states of a slowly varying Hamiltonian is an invariant. Furthermore, and perhaps most remarkably, entropy is an adiabatic invariant in the context of equilibrium thermodynamics.

The most well-known example goes back to the Lorentz–Einstein conversation during the Solvay conference (1911) – how the amplitude of a simple pendulum would vary if its period were slowly changed by shortening the string?
Box 1. A Simple Example

Let us consider a point particle moving in a rectangular potential well of width \( \ell \), colliding elastically with the wall. Allowing \( \ell \) to change slowly with time, we would like to find the change in energy.

The velocity changes by \( 2 \dot{\ell} \) after the particle with velocity \( v \) has collided with both the walls. Choose an interval of time \( \Delta t \) such that \( \Delta t \gg 2\ell/v \) and \( \Delta t \ll \ell/\dot{\ell} \). It is because of the slowness of the change of \( \ell \) that such a time interval exists. During this interval, there are \( v\Delta t/2\ell \) pairs (because of both walls) of collisions. Thus the velocity is changed by

\[
\Delta v = -v\dot{\ell} \frac{\Delta t}{\ell}.
\]

This implies that \( v\ell = \text{constant} \), or, \( E\ell^2 \) is “constant” – this is not an absolute constant value, only an adiabatic invariant. That is, it does not change significantly during a pair of collisions.

Einstein’s reply was that \( E/\omega \) will remain constant. It was (Paul) Ehrenfest who proved that the reply of Einstein is connected to the adiabatic invariance of action \[3\]. Let us consider the Hamiltonian of an oscillator in terms of the linear momentum \( p \) and position coordinate \( q \) of the mass \( m \):

\[
H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = \text{energy, } E, \tag{1}
\]

where \( \omega \) is the frequency of the oscillations. The \((q,p)\)-plane is called the phase plane or the phase space. For fixed energy \( E \), and given initial conditions, the curve traced by the phase space point with time is an ellipse. Solving for \( p(q,E) \), we can write

\[
J = \frac{2}{\pi} \int p(q,E) dq
= \frac{2}{\pi} \int_0^{\sqrt{2E/m\omega^2}} \sqrt{2mE - m^2\omega^2 q^2} \, dq = \frac{E}{\omega}. \tag{2}
\]

We can think in terms of a new transformed Hamiltonian \( \overline{H} = \omega J \). Imagine that there was a change of variables from \((q,p) \rightarrow (\theta, J)\), keeping in mind that in a proper place and momentum variables may be transformed to angle and action variables in phase space. Under slow changes, action is an adiabatic invariant. It is a quantity that resists change under slow evolution of the Hamiltonian; eventually, it must change, so it jumps.
transformation two independent variables must go over to two other independent variables. Calling \( \theta \) as the angle variable, and \( J \) as action, we have a new set of variables. We see that the transformed Hamiltonian, \( \mathcal{H} \), is independent of the angle variable. With this Hamiltonian as a function of \( \theta, J \), the equations of motion are

\[
\frac{d\theta}{dt} = \frac{\partial \mathcal{H}}{\partial J}, \quad \frac{dJ}{dt} = -\frac{\partial \mathcal{H}}{\partial \theta}.
\]

(3)

Clearly, \( \frac{dJ}{dt} = 0 \) (because \( \mathcal{H} \) is independent of \( \theta \)), and thus \( J \) is a constant.

If the length of the pendulum is varied slowly, the motion will remain oscillatory with an imperceptible change in frequency over a single oscillation. The action for this case may be written as an integral of \( p \) with respect to \( q \) over an unperturbed oscillation (justified because the change is so small):

\[
J' = \oint p\,dq = \oint \left( m\frac{dq}{dt} \right) \left( \frac{dq}{dt} \right) \,dt = \oint (2 \times \text{kinetic energy}) \,dt.
\]

(4)

The simple harmonic motion is given by sinusoidal functions. For instance, we may write the oscillatory motion as

\[
q(t) = q_0 \cos(\omega t + \delta),
\]

(5)

where the initial conditions on position and velocity will specify the constants, \( q_0 \) and \( \delta \). The potential energy is a quadratic function of \( q \) and kinetic energy is a quadratic function of \( dq/dt \). Both \( q^2 \) and \( (dq/dt)^2 \) are squares of sinusoidal functions, respectively square of cosine and sine according to the above solution. The integrals of both, kinetic and potential energies, over a single period of a harmonic oscillator are equal, and they must be individually equal to half of the total energy, \( E \). Thus,
(4) becomes

\[ J' = (2 \times \text{time average of kinetic energy}) \times (\text{time period, } T) = ET. \quad (6) \]

As the length of the string is slowly altered, the energy and frequency both change but their ratio is an adiabatic invariant.

Corrections to action which is an adiabatic invariant have been found [4]. A general proof of adiabatic invariance of action can be found in [5].

2. Introduction to Billiards with Two Degrees of Freedom

Dynamical systems with two degrees of freedom present two possible frequencies. We shall try to make this clearer in the following discussion by considering an example. In this section, there is no explicit dependence on time. Simple mechanical systems with one particle confined in a rectangular box (Figure 1), moving freely inside and reflecting from the boundary according to Snell’s law present a situation where there are two degrees of freedom. There are two constants of the motion. In rectangular billiard, the constants of the motion are the absolute values, or the squares, of the two components of linear momentum. (Note that reflection at a wall changes the sign of one component of the momentum but does not change its magnitude). Beginning with a given direction, there can be at most four possible directions (Figure 2). A trajectory inside a rectangle can be unfolded on a plane, as seen in Figure 3. Also,
Figure 5. The two circuits used in defining $I_x, I_y$ are seen here. On an appropriate identification of sides, these correspond to a toroidal and a poloidal circle*.

* When a rectangle has opposite sides glued together, it becomes a torus, and the two kinds of closed circuits, denoted by P and T in the figure below, are called poloidal and toroidal.

we note from Figure 4 that the part AB is identical to the segment beginning with E. Copies I and I’ are identical. Thus, we are obliged to sew the four copies I, II, III, IV, identifying AA’ with BB’, and, AB with A’B’. The surface thus obtained is two dimensional, topologically equivalent to a two-dimensional torus (bicycle tyre and medu wada being other examples of objects with this shape).

We can describe the dynamics in terms of a new set of variables where the new momentum variables are the conserved quantities. We would like to do this, as it will make the description very simple. To effect this, we write two integrals (Figure 5):

$$I_x = \oint_{\Gamma_1} p_x \, dx = p_x \cdot 2a \quad \Rightarrow \quad p_x = \frac{I_x}{2a},$$

$$I_y = \oint_{\Gamma_1} p_y \, dy = p_y \cdot 2b \quad \Rightarrow \quad p_y = \frac{I_y}{2b}.$$  \hspace{1cm} (7)

The Hamiltonian is

$$H = \frac{p_x^2 + p_y^2}{2m} = \frac{1}{8m} \left[ \frac{I_x^2}{a^2} + \frac{I_y^2}{b^2} \right].$$  \hspace{1cm} (8)

$I_x, I_y$ are called action variables. We can write the Hamilton’s equations and get the frequencies:

$$\omega_x = \frac{d\theta_x}{dt} = \frac{\partial H}{\partial I_x} = \frac{I_x}{4ma^2} \quad \Rightarrow \quad \theta_x(t) = \theta_x(0) + \frac{I_x}{4ma^2}t,$$

$$\omega_y = \frac{d\theta_y}{dt} = \frac{\partial H}{\partial I_y} = \frac{I_y}{4mb^2} \quad \Rightarrow \quad \theta_y(t) = \theta_y(0) + \frac{I_y}{4mb^2}t.$$  \hspace{1cm} (9)

Moreover,

$$\frac{dI_x}{dt} = -\frac{\partial H}{\partial \theta_x} = 0, \quad \frac{dI_y}{dt} = -\frac{\partial H}{\partial \theta_y} = 0,$$

implying that $I_x, I_y$ are constants of the motion.
3. Deeper into the Two-Dimensional Jungle

The rectangular billiard discussed above has two degrees of freedom, two independent constants of the motion, the motion is smooth everywhere, the surface on which the trajectory resides is a two-dimensional torus. Such a system is an example of an integrable system. The dynamics is easy to describe. The constants of the motion are clearly due to the symmetries of the system. Notice that if one of the sides of the rectangle is chopped to make it slanting at an angle, the two components of the linear momentum of the particle will get mixed on reflection. However, this kind of perturbation is not mathematically smooth. What we have in mind here are ‘mathematically smooth’ perturbations – for instance, a slow rotation of the rectangle about its centre with a particle moving inside freely. A nonlinear pendulum, with equation of motion $\ddot{q} + \sin q = 0$, when subjected to a periodic perturbation, gives us a rather interesting new equation: $\ddot{q} + \sin q = -\epsilon k \sin(kq - \Omega t), \epsilon \ll 1$. With one space and a time coordinate, we have two frequencies once again. This equation is encountered when describing the motion of a charged particle in the field of two electrostatic plane waves, destruction of magnetic surfaces in a tokamak by resonant magnetic perturbations, and a charged particle trapped in a toroidal magnetic field but perturbed by a superposed electrostatic wave [6].

Generically, we arrive at a perturbation of a regular (read integrable) system, where the total Hamiltonian can be written as

$$H(J_1, J_2, \theta_1, \theta_2) = H_0(J_1, J_2) + \epsilon H_1(J_1, J_2, \theta_1, \theta_2). \quad (11)$$

The unperturbed Hamiltonian is solvable. Since $H_1$ is a periodic function of $\theta_1, \theta_2$, we may expand it in a Fourier series:

$$H_1 = \sum_{n_1, n_2} H_{n_1, n_2}(J_1, J_2) \exp[i(n_1 \theta_1 + n_2 \theta_2)]. \quad (12)$$

The constant energy (or invariant) surface of an integrable system in phase space is torus-shaped. Smooth perturbations of an integrable system create significant changes in phase-space regions where the unperturbed frequencies become commensurate.
The two unperturbed frequencies could be commensurate with each other, or be in resonance, called as non-linear resonance [7, 8]:

\[ s\omega_x + r\omega_y = 0, \quad \text{for some integers } r, s. \] (13)

The resonance condition involves phase space variables (e.g., \( I_x, I_y \) in the example considered in Section 2). This means that the resonance condition will affect this region of the phase space. Away from the resonance, \( I_x \) and \( I_y \) are constants. How does the dynamics modify near the resonance? Let us call the action \( I = I^r \) the resonant action. For a simple illustration, we approximate the perturbation neglecting weak high-frequency effects:

\[ H(I, \theta) = H_0(I) + \epsilon H_1(I) \cos(s\theta_x + r\theta_y). \] (14)

The resonance, close to resonant action, is characterized by

\[ s\omega_x(I^r) + r\omega_y(I^r) = 0. \] (15)

Close to the resonance, where the phase space modification is expected to occur, the equations of motion are

\[
\begin{align*}
\frac{dI_x}{dt} &= -\frac{\partial H}{\partial \theta_x} = \epsilon sH_1(I) \sin(s\theta_x + r\theta_y), \\
\frac{dI_y}{dt} &= -\frac{\partial H}{\partial \theta_y} = \epsilon rH_1(I) \sin(s\theta_x + r\theta_y).
\end{align*}
\] (16)

Multiplying the first equation by \( r \) and the second by \( s \) and subtracting

\[
\frac{d}{dt}(rI_x - sI_y) = 0 \implies rI_x - sI_y = \text{constant of the motion}. \] (17)

We can now see that \( s\theta_x + r\theta_y \) is a special coordinate. We can define new coordinates as \( J \) and \( \psi \):

\[
\begin{align*}
\psi_1 &= s\theta_x + r\theta_y, \quad \psi_2 = \alpha\theta_x + \beta\theta_y, \\
J_1 &= [\beta(I_x - I^r_x) - \alpha(I_y - I^r_y)]/(s\beta - r\alpha), \\
J_2 &= [r(I_x - I^r_x) - s(I_y - I^r_y)]/(-s\beta + r\alpha).
\end{align*}
\] (18)
where $\alpha, \beta$ are constants such that $s\beta - r\alpha \neq 0$. With this, (14) becomes simply the resonant Hamiltonian

$$H_1(J, \psi) = H_0(J) + \epsilon H_1(J) \cos \psi_1. \tag{19}$$

This is independent of $\psi_2$; thus, $dJ_2/dt = -\partial H_1/\partial \psi_2 = 0$, implying that $J_2$ is a constant of the motion. To investigate the motion near the nonlinear resonance, we can expand $H_0(J)$ in a Taylor series in two variables about the origin (as we are now sitting on the resonance):

$$H_0(J) = H_0(0) + \frac{\partial H_0}{\partial J} \bigg|_{(J_1, J_2) = (0, 0)} J + \frac{1}{2!} \left[ \frac{\partial^2 H_0}{\partial J_1^2} \bigg|_{(J_1, J_2) = (0, 0)} J_1^2 + 2 \frac{\partial^2 H_0}{\partial J_1 \partial J_2} \bigg|_{(J_1, J_2) = (0, 0)} J_1 J_2 \right. \left. + \frac{\partial^2 H_0}{\partial J_2^2} \bigg|_{(J_1, J_2) = (0, 0)} J_2^2 \right]. \tag{20}$$

Since $J_2$ is a constant of the motion and does not appear in (19), we may set it to zero. This simplifies (20):

$$H_0(J) = H_0(0) + \frac{\partial H_0}{\partial J} \bigg|_{(J_1, J_2) = (0, 0)} J + \frac{J_1^2}{2!} \frac{\partial^2 H_0}{\partial J_1^2} \bigg|_{(J_1, J_2) = (0, 0)} \tag{21}$$

The second term is simply

$$\frac{\partial H_0}{\partial J} \bigg|_{(J_1, J_2) = (0, 0)} J = \frac{\partial H_0}{\partial J_1} \bigg|_{J_1 = 0} = \frac{\partial H_0}{\partial I_x} \bigg|_{I_1 = I_r} \frac{\partial I_x}{\partial J_1} \bigg|_{J = 0} + \frac{\partial H_0}{\partial I_y} \bigg|_{I_1 = I_r} \frac{\partial I_y}{\partial J_2} \bigg|_{J = 0} \tag{22}$$

$$= \omega_x (I') s + \omega_y (I') r = 0 \tag{23}$$

by the resonance condition. Thus (19) is

$$H(J, \psi) = H_1^r(J, \psi) \quad = H_0(0) + J_1^2/2 [\frac{\partial^2 H_0}{\partial J_1^2} \bigg|_{J_1 = 0}]^{-1} + \epsilon H_1(J = 0) \cos \psi_1. \tag{25}$$

In (25), the resonant Hamiltonian has a rotational kinetic energy term and a potential energy term which is a cosine function, just like a nonlinear pendulum. Around each resonance, there will be such a modification. You may imagine what happens if these regions begin to overlap.
The dynamics in the vicinity of a resonance is given by the resonant Hamiltonian (25) – which is exactly the Hamiltonian of a pendulum! The phase space trajectories are shown in Figure 6.

The discussion presented above will remain unchanged if the perturbation (14) is modified to include time dependence, i.e.,

\[ H(I, \theta) = H_0(I) + \epsilon H_1(I) \cos(s\theta + r\Omega t). \]  

The important point here is that for a nonlinear resonance we need at least two degrees of freedom or one degree of freedom and an explicit time dependence.

In either case, the perturbed system will present two situations – motion away from the resonance and near the resonance. Away from the resonance, the motion is similar to the unperturbed system. Near the resonance, the dynamics can be described with suitable variables by the Hamiltonian of a pendulum. Therefore, the phase space in these variables can be understood in terms of a nonlinear pendulum including the presence of a separatrix (Figure 6).

To see what happens close to the separatrix (the boundary of ‘close’ and ‘far’ from a nonlinear resonance), let us note first that (1) can be obtained from

\[ H = \frac{p^2}{2m} - m\omega^2 \cos q \]  

by expanding the cosine in a Taylor series about \( q = 0 \). We can ignore the constant term in the expansion. From \( dq/dt = \partial H/\partial p \),

\[
\begin{align*}
  t & = \int \frac{dq}{\partial H/\partial p}, \text{ and the complete period is} \\
  T & = \sqrt{\frac{m}{2}} \int \frac{dq}{\sqrt{E + m\omega^2 \cos q}},
\end{align*}
\]

Here we try to get closer to understanding the critical phase curve, the separatrix.
Figure 6. The phase space portrait of a nonlinear pendulum. The horizontal axis represents angle measured from the lowest point, and the vertical axis the corresponding angular momentum, proportional to the angular velocity. The closed curves near the origin correspond to small oscillations. After the pendulum is released from a finite amplitude at positive $q$, the particle starts moving with negative angular velocity, towards the origin. Thus the closed curves are traversed clockwise. If the pendulum is given a sufficiently high positive angular velocity to start with, then it reaches the top of the circle (a single point, represented by $\pm \pi$ in the diagram), and continues moving in the same direction. This kind of motion is described by the curves running from left to right in the upper part of the diagram. Circulation in the opposite sense is described, similarly, by the curves running from right to left at the bottom of the diagram. The curve separating these two kinds of motion is called the ‘separatrix’. The upper part corresponds to a particle starting from the top in the infinite past, and falling to the right, and reaching the top from the left in the infinite future. The lower part corresponds to falling to the left from the top, which is a point of unstable equilibrium, represented by the point with zero angular momentum, and angular co-ordinate equal to $\pm \pi$.

which is an elliptic integral. The closed orbits around the origin in Figure 6 correspond to small oscillations (called librations). These are then surrounded by other closed orbits which are called oscillations. Then, we notice open curves running from left to right, or right to left, outside – these correspond to rotations. These two distinct motions are separated by the curve called the
separatrix. The origin is a type of point, called an elliptic point. The points $\pm \pi$ are seen to be approached by vector fields along one direction. There is another direction along which the vector fields are diverging away. These points are saddle points and present the positions of unstable equilibria. The separatrix is obtained from (27) with the separatrix condition $E = m\omega^2$. We can write for the momentum for separatrix as [9]

$$ p_{sx} = \pm \sqrt{2} m \omega \sqrt{1 + \cos q_{sx}} = \pm 2 m \omega \cos \frac{q_{sx}}{2}; \quad (29) $$

the two signs correspond to the upper and lower branches in Figure 6. We have

$$ \frac{dq_{sx}}{dt} = \frac{p_{sx}}{m} = \pm 2 \omega \cos \frac{\phi_{sx}}{2}. \quad (30) $$

Integrating (30) over time with $q = 0$ at $t = 0$, we get

$$ \omega t = \int_0^{q_{sx}} \frac{dq/2}{\cos q/2} = \log \tan \left( \frac{q_{sx}}{4} + \frac{\pi}{4} \right) \quad (31) $$

or

$$ q_{sx} = 4 \tan^{-1}[\exp(\omega t)] - \pi. \quad (32) $$

This discussion on the pendulum gives the precise nature of trajectories that develop around a nonlinear resonance.

We have seen that $J_2$ is a constant of the motion. So, the dynamics can be expressed in terms of $J_1$ and $\psi_1$. For instance, beginning with the point A, one reaches a point B after going around the torus (Figure 7). The

Figure 7. The invariant surface is a torus, a section of which is seen in terms of the variables $J_1, \psi_1$. Beginning at a point A on the surface of the section, the trajectory goes around and pierces the section at B. On the section, the map takes the point A to B by a twist map.
successive points are given by a map on a section of a pair of coordinates,

\[
\begin{align*}
J_{n+1} &= J_n, \\
\psi_{n+1} &= \psi_n + 2\pi \alpha(J_{n+1}), \quad \alpha = \omega_1/\omega_2,
\end{align*}
\]  
(33)

where we have dropped the subscript ‘1’. If \( \alpha \) is irrational, then the trajectory fills the circle uniformly as \( n \to \infty \). Once we have perturbed the integrable system, on the surface of section where \( \theta_2 \) is constant (mod \( 2\pi \)), the twist map (33) changes to perturbed twist mapping [9]

\[
\begin{align*}
J_{n+1} &= J_n + \epsilon f(J_{n+1}, \psi_n), \\
\psi_{n+1} &= \psi_n + 2\pi \alpha(J_{n+1}) + \epsilon g(J_{n+1}, \psi_n),
\end{align*}
\]  
(34)

where \( f \) and \( g \) are periodic functions of angle variables. The fact that this map is area preserving (Liouville theorem) implies that

\[
\frac{\partial f}{\partial J_{n+1}} + \frac{\partial g}{\partial \psi_n} = 0.
\]  
(35)

The functions \( f \) and \( g \) can be found by going back to the Hamilton’s equation,

\[
\frac{dJ_1}{dt} = -\epsilon \frac{\partial H}{\partial \psi_1}.
\]  
(36)

We can integrate over one period of the motion along \( \psi_2 \) to get the change in \( J_1 \) as

\[
\Delta J_1 = -\epsilon \int_0^{T_2} \frac{\partial H_1}{\partial \psi_1}(J_{n+1}, J_2, \psi_n + \omega_1 t, \psi_{20} + \omega_2 t) dt,
\]  
(37)

where \( \omega_i = 2\pi/T_i, \quad i = 1, 2 \). Using unperturbed values, we can integrate \( \partial H_1/\partial \psi_1 \) to get the jump in action, \( \Delta J_1 \) from (34), equal to \( \epsilon f \). The jump in phase can be found by using Liouville theorem (35) in (34):

\[
g(J, \psi) = -\int_{\psi}^{\psi'} \frac{\partial f}{\partial J} d\psi'.
\]  
(38)
The separatrix is a pair of curves held on the two ends by saddle points at $\pm \pi$ and $-\pi$. This reminds us of a popular game where a player skips the rope as it is moved around by the two players on either side. The crossing of the separatrix is analogous to the jumping player stepping on the rope, and getting out in the game.

We have now described the motion near a nonlinear resonance. Let us use these ideas to calculate the change in adiabatic invariant across the separatrix, i.e., as the system crosses an instability.

4. Jump in Action Across the Separatrix: The Melnikov–Arnold Integral

We should keep in mind that the motion near a nonlinear resonance is modified according to a pendulum Hamiltonian (equation (25)). There are two unstable (saddle) points, $\psi = \pm \pi$. The instability in the motion is created in the vicinity of these points. If we were to place a ‘drop’ of initial conditions centered at $\psi = \pi$ (or $-\pi$), on further iterations, this ‘drop’ will contract along one direction and expand in the other direction. Eventually, it will become a thin, long fibre folding all over without intersecting ever. This is the reason that it is very important to be able to calculate the change in action across a separatrix. On the two sides of the separatrix, there are two sets of phase space curves labelled by different values of the invariant on either side. The value that labels these curves, inside and outside, jumps. The trajectory of the system corresponding to the ‘outside’ curve is untrapped in contrast to the one inside the separatrix.

Returning to our discussion at (12), we can now write the total Hamiltonian in the new coordinates:

$$H = H_0(J) + \frac{p^2}{2m} - \epsilon m \omega^2 \cos \phi$$

$$+ \epsilon \sum_{p>1,q \neq 0} \Lambda_{pq} \cos \left(\frac{p}{r} \phi - \frac{q}{r} \theta + \chi_{pq}\right), \quad (39)$$

where $\Lambda$ and $\chi$ are functions of $J$. The frequency associated with $\theta$ will change, even as a function of $J$. For a moment, we may imagine this to be an oscillator Hamiltonian which is being driven, and imagine $\theta$ to be $\Omega t$ for some fixed $\Omega$. Dropping $H_0(J)$, we can now rewrite an
equivalent Hamiltonian of a driven pendulum,

\[ H' = \frac{p^2}{2m} - \epsilon m \omega^2 \cos \phi \]
\[ + \epsilon \sum_{p>1, q \neq 0} \Lambda_{pq} \cos \left( \frac{p}{r} \phi - \frac{q}{r} \Omega t + \chi_{pq} \right). \] (40)

We can see that the (action) variable conjugate to \( \Omega t \) is \( J = H' / \Omega \) (recall the discussion leading to (2)). Hence, \( \Delta J = -\Delta H' / \Omega \). We now drop \( \Lambda_{pq} \) except for \( p = q = 1 \), and call \( \Lambda_{11} \) by \( \Lambda \) assuming that the high-frequency terms are weaker. We also assume that the frequency of small librations is much smaller than \( \Omega \). Using (37), we get

\[ \Delta J = -\epsilon \Lambda_{11} \int_{-\infty}^{\infty} dt \sin \left[ \frac{1}{r} \{ \phi(\omega t) - (\Omega t + \theta_n) \} \right]. \] (41)

The unperturbed separatrix motion is in \( \phi(\omega t) = 4 \tan^{-1} e^{\omega t} - \pi \), the angle \( \theta \) is \( (\Omega t + \theta_n) \). The phase of \( \theta \) at the \( n \)th crossing of the surface of section \( \phi \simeq \pm \pi \) is \( \theta_n \), where \( \chi \) has been included in it. Let \( s = \omega t \). Expanding the sine function above in two terms, we see that the term \( \sin[\phi - (\Omega t)/r] \cos \theta_n \) is odd in \( t \) under symmetric limits, and thus contributes nothing. The other term with \( \sin \theta_n \) contributes:

\[ \Delta J = \frac{\epsilon \Lambda}{\omega_0 r} \sin \theta_n \int_{-\infty}^{\infty} ds \cos \left[ \frac{1}{r} \phi(s) - \frac{s}{r} \Omega \right] \]
\[ := \frac{\epsilon \Lambda}{\omega r} \sin \theta_n \mathcal{I}_{MA}, \] (42)

where \( \mathcal{I}_{MA} \) is called the Melnikov–Arnold integral. This integral is improper, with an oscillating part and a jump. The oscillating part is, in fact, larger than the jump but it averages to zero. We will not get into the details of the calculation of this integral. The details can be seen in [11]. However, just to get a glimpse of what we are talking about, we show the plot of the integral, \( \mathcal{I}_{MA} \) as a function of \( \Omega / r \omega \) (Figure 8).

*The change in adiabatic invariant (action) is made very complex due to the highly unstable nature of the separatrix. The jump in action is accompanied by violent oscillations. This complexity is succinctly captured by the Melnikov–Arnold integral.*
5. Summary

It is important to understand the concept of adiabatic invariance [10]. In simple words, if we imagine that a particle is subjected to slow perturbation, the curves in (position, momentum)-space (phase space) change their shape compared to the unperturbed physical situation. Further, since work is being done in effecting this change, the curves will deviate from the unperturbed case. However, the new curves are such that the area enclosed by them is the same as before. This area is ‘action’. We have tried to bring out the behaviour of an adiabatic invariant across an unstable equilibrium of a nonlinear pendulum Hamiltonian. As the curve approaches the point of unstable equilibrium, the separatrix, where the time period of the oscillation approaches infinity (and frequency approaches zero), the adiabatic invariant becomes less accurate as a label of the curve. And, as the separatrix is crossed, this invariant jumps to a larger value. This is because the area accessible to the curve is due to the two segments, upper and lower halves (Figure 6).

The remarkable point is that the dynamics near a nonlinear resonance is described by a Hamiltonian that corresponds to a nonlinear pendulum. The nonlinear pendulum Hamiltonian corresponds to the slow variables,
moving in the frame of resonance frequency. The separatrix of this pendulum is perturbed by the remaining ‘fast’ terms. This transforms the separatrix into a chaotic layer. Instances of this generic situation abound in physics literature.

We have only focussed on systems with two degrees of freedom. Three-dimensional or time-dependent two-dimensional versions of this discussion will lead us to another important contribution of Arnold, called the Arnold diffusion. Here the chaotic layers of different resonances overlap and make a web, the Arnold web. This facilitates slow diffusion. These thread-like regions which make a web help in transport of energy. The author hopes that the introduction presented here will encourage the interested readers to explore the subject in greater detail in more advanced references (for instance, [12, 13, 14]). The application of these ideas to quantum transport in the context of semiconductor superlattices subjected to electric and magnetic fields is of significance (a readable account may be found in [15]).

**Suggested Reading**