On Embedding a Semigroup in a Group

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In this article, we explain the idea of Rees [1] to utilize the inverses of partial one-one functions on a cancellative monoid S to embed S into a group.

1. Introduction

The development of mathematics could be seen to have started with embedding theorems. Embedding the system of natural numbers into the ring of integers, embedding the ring of integers into the field of rational numbers, embedding the rational field into the field of real numbers and embedding the real number system $\mathbb{R}$ into the complex number field constitute an inspiring saga of classical mathematics. Embedding the algebra $\mathbb{C}$ of complex numbers into the skew field $\mathbb{H}$ of quaternions, embedding the algebra of quaternions in the (non-associative) alternative algebra $\mathbb{O}$ of octonions and concluding that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only finite dimensional real algebras is a success story of the 20th century.

Development of mathematics may be said to have started with embedding theorems.

In the present article, we wish to discuss the problem of embedding a semigroup in a group. A semigroup is among the simplest algebraic objects one could imagine. Its very simplicity made many people skeptical about the significance of semigroup theory. Commenting on the attitude of many mathematicians in the 1950's, the famous semigroup theorist Preston [2] says, "I became aware that some mathematicians regarded semigroup theory, I think it is not too strong to say, with scorn, because its results were too superficial, only when I began talking to others working principally in semigroup theory. I have been told, since coming to this confer-

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ence, that such a view prevailed for many years with many Russian mathematicians". Being a simple object with not too much of a structure makes a semigroup an ideal vehicle to introduce the challenge and thrill of mathematical problems.

Semi group theory is not trivial as was thought by some.

1.1 *What are Semigroups*?

We recall that a pair (G, o) is called a semigroup if G is a nonempty set and 'o' is an associative binary operation on G. The most natural example of a semigroup is the set $M(A)$ of all selfmaps $f : A \to A$ of a nonempty set A with composition of mappings as the binary operation. Another natural example is given by the set M_n of all $n \times n$ real matrices with matrix multiplication as the binary operation. A semigroup (G, o) is called a monoid if G contains an identity element 'e' satisfying $aoe = eoa = a$ for all a in G. A monoid (G, o) is called a group if to each a in G there corresponds an element a^{-1} in G such that $aoa^{-1} = a^{-1}oa = e$. In a monoid, identity elements are unique. If G is a group and $a \in G$, the element a^{-1} satisfying $aoa^{-1} = a^{-1}oa = e$ is unique. This unique element is called the inverse of a. The semigroups $M(A)$ and M_n are monoids but not groups. The set of even integers is a semigroup under multiplication but is not a monoid. The set of nonzero integers is a monoid but not a group under multiplication. A subset A of a semigroup G is called a right ideal of G if $AG \subseteq A$, where $AG = \{ag/a \in A, g \in G\}$.

1.2 *Defining Semigroups by Generators and Relations*

Let us now briefly describe the notions of free semigroups and defining relations. Let A be a nonempty set and let $F(A)$ consist of all finite strings of the form $a_1a_2...a_n$, where n is a positive integer and $a_1, a_2, ..., a_n$ are all members of A. Then $F(A)$ is a semigroup if we define the binary operation 'juxtaposition' by $(a_1a_2...a_n)$ $(b_1b_2...b_m) = a_1a_2...a_nb_1b_2...b_m$. Note that $F(A)$ is a

Defining a semigroup by generators and relations.

semigroup but not a monoid. $F(A)$ is called the free semigroup generated by A. Let S denote $F(A) \cup \{\epsilon\}$, where ϵ is an element not in $F(A)$. Define $(a_1a_2...a_n)\epsilon = a_1a_2...a_n$, $\epsilon(a_1a_2...a_n) = a_1a_2...a_n$ and $(a_1a_2...a_n)(b_1b_2...b_m) = a_1a_2...a_nb_1b_2...b_m$ for $a_1a_2...a_n, b_1b_2...b_m$ in A. Then S is a monoid called the free monoid generated by A. The monoid S is free in the sense that all imaginable products of elements of A are available in S without any restrictions or conditions.

Now suppose we want a monoid that satisfies some relations between some products of elements of A. If A denotes the English alphabet, then S is the set of all possible English words (both meaningful and meaningless) together with the 'empty word' ϵ. Suppose, instead of the semigroup of all possible English words, we wish to construct a monoid consisting of all English words but with the restriction that the strings 'color' and 'colour' are not to be distinguished wherever they occur. But then the strings 'tricolor' and 'tricolour' should be treated as identical too. Also, the strings 'colorful' and 'colourful' need to be identified as the same. So, as soon as we wish a relation $x = y$ to be true, we should also impose the relations $ax = ay$ and $xa = ya$ for all a in S. Let R consist of all the pairs (x, y), the coordinates of which are meant to be indistinguishable. It is natural to wish that (a) each element x is indistinguishable from itself, (b) if x is indistinguishable from y then y is indistinguishable from x and (c) if y is indistinguishable from both x and z, then x and z are also indistinguishable from each other. So, our original wish that the pairs of R consist of indistinguishable elements leads to a possibly bigger set $\overline{R}$ consisting of pairs of indistinguishable elements. Our discussion above shows that $\overline{R}$ is an equivalence relation on S and that $(x, y) \in \overline{R}$ should imply $(ax, ay) \in \overline{R}$ and $(xa, ya) \in \overline{R}$ for all x in S. Such equivalence relations are called congruence relations. Naturally, we want the smallest congruence relation $\overline{R}$ such that $R \subseteq \overline{R}$. Now $S \times S$ is the biggest congruence relation on S such that $R \subseteq S \times S$. The intersection of all the congruence relations $R^{'}$ on S such that $R \subseteq R^{'}$ is the smallest congruence relation on S which contains R as a subset. Hence $\overline{R} = \cap R^{'}$, where the intersection is over all the congruence relations on S that contain R. Since $\overline{R}$ is an equivalence relation, it is meaningful to talk about $S/\overline{R}$. If $[a]$ and $[b]$ are two elements of $S/\overline{R}$ then the definition $[a][b] = [ab]$ can be easily verified to be a proper definition. This operation in $S/\overline{R}$ makes $S/\overline{R}$ into a monoid called the semigroup on A generated by the set of defining relations R.

1.3 *The Embedding Problem – History*

The problem of embedding a semigroup in a group remained an enigma for quite long. It is observed that if a semigroup $(S, \cdot)$ is embeddable in a group $(G, \cdot)$ then S should be cancellative: $ax = bx \Longrightarrow a = b$ and $xa = xb \Longrightarrow a = b$. So, the problem to be considered is to embed a cancellative semigroup in a group. If S is commutative besides being cancellative then it can be embedded in a group by a process identical to that of embedding an integral domain in a field.

Anton Kazimirovich Sushkevich (1889–1961) [3] gave an erroneous proof that any cancellative semigroup can be embedded in a group. Christopher Hollings says, "Sushkevich was a Russian born mathematician, educated in Berlin and St. Petersburg, who spent most of his working life at Kharkov State University in the Ukraine. He initiated the systematic study of various classes of 'generalized groups' (sets with binary operations, subject to certain other conditions), amongst which we may now recognize both semigroups and loops. Unfortunately, Sushkevich's work is not particularly well known, especially in the West. From 1933 until his death, Sushkevich was head (indeed, the first head) of the Cathedra of Algebra and Number Theory (later the

Historical origins of the problem. Malcev proves Sushkevich wrong.

Sushkevich's obstinacy.

Cathedra of Algebra and Mathematical Logic) within the Mathematics Department of Kharkov State University. He was the author of a respected undergraduate algebra textbook, as well as one on number theory, and also had an interest in the history of mathematics". Malcev gave an example to show that there are cancellative semigroups which cannot be embedded in a group. Commenting on Sushkevich's persistence with his erroneous ideas even after being proved wrong by Malcev and even after adverse criticism from his peers, Christopher Hollings has written about 'The Perils of Taking Shortcuts' in [4]. This should convince the reader that objectivity is an ideal to be practiced assiduosuly and neglect of it is perilous.

However, in 1935, Sierpinski showed that every countable semigroup is isomorphic to a semigroup of self-mappings on $\mathbb{N}$; this was used by Banach to deduce that every countable semigroup can be embedded in a two-generated semigroup.

1.4 *Van der Waerden's Question*

Before describing Malcev's example, it is worth mentioning that the famous algebraist van der Waerden asked if every non-commutative ring without zero-divisors can be embedded in a skew-field. Malcev used his example to answer this question in the negative. In fact, Malcev realized that once he obtained a semigroup $(G, .)$ that could not be embedded in a group, he could then use it to construct a commutative ring without zero-divisors that cannot be embedded in a skew-field in the following way. Let R consist of all expressions of the form $\sum k_i x_i$, where the k_i are rational numbers and only finitely many of them are nonzero and the x_i are elements of G, and define addition and multiplication in R on expected lines. Then R is a non-commutative ring without zero-divisors. It is clear that R cannot be embedded in a skew-field because $(G, .)$ cannot be embedded in a group.

Malcev realizes that the essential answer to van der Waerden's question lies in constructing a semigroup not embeddable in a group.

1.5 *Malcev's Example*

Consider the semigroup S generated by the symbols a, b, c, d, x, y, u, v subject to the defining relations $ax = by, cx = dy, cu = dv$. In S, the empty word serves as the identity. S is clearly cancellative. If S were embeddable in a group then we would have $a^{-1}b = xy^{-1}, xy^{-1} = c^{-1}d$ and $c^{-1}d = uv^{-1}$ from the defining relations. But then $a^{-1}b = uv^{-1}$ and so $au = bv$, a contradiction since in the word au there is no occurrence of ax or by or cx or dy or cu or dv so that it could be transformed into bv and we could say the same about bv. Malcev's example may be found, for example, in [3].

1.6 *Ore's Condition*

Oystein Ore [3] imposed an additional condition on a cancellative monoid to make it embeddable in a group.

A semigroup $(S, \cdot)$ is said to satisfy Ore's condition or is said to be left reversible if to each a, b in S, there correspond c, d in S such that $ac = bd$. In other words, $aS \cap bS \neq \emptyset$.

Remark: Ore's condition may be stated as "Different elements of S possess at least one common right multiple." It is pertinent to remark that Ore's condition is true in a commutative semigroup since in that case $ab \in aS \cap bS$.

1.7 *Motivating the Embedding Theorem*

Before proving our embedding theorem it is useful to make some motivational remarks.

Suppose we start with the monoid $(W, +)$ where $W = \{0, 1, 2, 3, ...\}$ and '+' is the usual addition. It is fairly obvious that an element, say, 3 of W is mimicked well by the function $f_3 : W \to W$ taking 0 to 3, 1 to 4, 2 to 5 and so on. Equally obviously, f_0 denotes the identity mapping on W. The problem now is to embed $(W, +)$ in a group $(G, +)$. Why is $(W, +)$ not a group? The answer is that it lacks additive inverses. Now look at 3 in W

Mighty oaks from little acorns grow!

again. f_3 is the function $f_3(x) = 3 + x$ with domain W. Moreover, it is a 1-1 function, i.e., an injection. Does it have an inverse? f_3 is not a bijection since it is not onto (for instance, the element 0 of W is not an f_3-image) and so the function f_3 does not have an inverse function. But f_3 is also a relation. In fact, $f_3 = \{(0,3),(1,4),(2,5),...\}$. What relation is the inverse of this relation? ${f_3}^{-1} = \{(3,0),(4,1),(5,2),...\}$. While ${f_3}^{-1}$ is not a function on W (it is not defined at the elements $0,1,2$ of W), it is a partial function of W. A function $g : A \to B$ is called a partial function on W if A, B are subsets of W. The partial function f_3 on W is a function on $A = \{3,4,5,...\}$ and it is 1-1 as a function from A into W. Such partial functions are called 1-1. In what follows, for any set A, i_A denotes the identity function on A.

For another example, consider the function $f : A \to B$ where $A = \{-1,-2,-3,...\}$ and $B = \mathbb{Z}$, the set of integers defined by $f(x) = x^2$. f is 1-1 on A since $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow -|x| = -|y|$. The range of f is $\{1,4,9,...\}$ and f is a partial function on $\mathbb{Z}$. The relation $f^{-1} = \{(1,-1),(4,-2),(9,-3),...\}$ is a function from $P = \{1,4,9,...\}$ onto $Q = \{-1,-2,-3,...\}$. Thus f^{-1} is also a partial function on $\mathbb{Z}$. Now $(f \circ f^{-1})(x) = i_P(x)$ for all x in P and $(f^{-1} \circ f)(x) = i_Q(x)$ for all x in Q. For the reason that $f \circ f^{-1}$ and $f^{-1} \circ f$ have different domains, we cannot write $f \circ f^{-1} = f^{-1} \circ f$. But we can salvage something from this.

We can write $f \circ f^{-1} = f^{-1} \circ f$ on $P \cap Q$ (in fact all statements are true about the elements of $P \cap Q$ as it is empty; it is empty of substance!). But look at our first example. There we can write $f_3 \circ {f_3}^{-1} = {f_3}^{-1} \circ f_3$ for all x in $P \cap Q$, where $P = W$ and $Q = \{3,4,5,...\}$. In this example, matters are not so hopeless as in the other example because the domains of f_3 and ${f_3}^{-1}$ are not disjoint. Later we shall see, in the proof of our theorem, how Ore's condition ensures that the common domain of two partial functions of interest to us is non-empty. In fact, the very idea and purpose of Ore's condition is to preclude the possibility of empty domains. Moreover, we may write $f_3 \circ {f_3}^{-1} = {f_3}^{-1} \circ f_3 = i_Q$ where $Q = \{3, 4, 5, ...\}$. We can also write $(f_3 \circ {f_3}^{-1})(x) = f_0(x)$ for all $x \in Q$.

We can make a similar statement about f_7 by writing $(f_7 \circ {f_7}^{-1})(x) = f_0(x)$ for all $x \in \{7, 8, 9, ...\}$. Further

(a) $(f_5 \circ {f_2}^{-1})(x) = f_5(x - 2) = x + 3 = f_3(x)$ for $x \in \{2, 3, 4, ...\}$,

(b) $({f_2}^{-1} \circ f_5)(x) = {f_2}^{-1}(x + 5) = x + 3 = f_3(x)$ for $x \in \{5, 6, ...\}$,

(c) $(f_{96} \circ {f_{93}}^{-1})(x) = f_{96}(x - 93) = x + 3 = f_3(x)$ for $x \in \{93, 94, ...\}$,

(d) $({f_{93}}^{-1} \circ f_{96})(x) = {f_{93}}^{-1}(96 + x) = x + 3 = f_3(x)$ for $x \in \{96, 97, ...\}$ and so on.

These examples (a),(b),(c),(d) and their ilk have a moral to convey. The example (a) almost says $5 - 2 = 3$, while the examples (b),(c) and (d) almost convey $-2 + 5 = 3, 96 - 93 = 3$. They hesitate to convey the same because of the difference in the domains. But we do wish $96 - 93 = 5 - 2$ to be true. If only we agreed to ignore the distinctions in domains (it is a small price to pay to turn the monoid $(W, +)$ into the group $(Z, +)$), then we would have created 'negative integers' starting from whole numbers. This wonderful idea of utilizing the inverses of 1-1 partial functions on a monoid $(S, \cdot)$ to create inverses for the elements of S by mathematically justifying the neglect of trivial distinctions and thereby obtain a group $(G, \cdot)$ containing $(S, \cdot)$ as a subgroup is due to Rees. As all students of mathematics know, the obliteration of inessential distinctions consists in finding a congruence relation θ and factoring it away. In what

Ends justify the means! Our goal of embedding necessitates Ore's conditions.

follows, *dom* f denotes the domain of the function f and *ran* f denotes its range.

Let us now see how Ore's condition is necessitated by our approach to the embedding problem through partial functions. As our discussion above shows, the group G is envisaged to be generated by the f_a's and their inverses. The f_a's have S as their domain and hence $f_a \circ f_b$ will be defined as a (nonempty) function for all a, b in S. Also the element baa of S is in $dom\ f_a^{-1} \circ f_b^{-1}$ and $f_a^{-1} \circ f_b^{-1}(baa) = a$ for all a, b in S. However, the case of ${f_a}^{-1} \circ f_b$ is not so simple. $dom\ f_a^{-1} \circ f_b \neq \emptyset$ iff $f_a^{-1} \circ f_b(x)$ is defined for some x in S iff $f_a^{-1}(bx)$ is defined for some x in S iff $bx = ay$ for some x, y in S. This is Ore's condition!

2. Main Result

Let us now get ready for some serious mathematics and see how, if at all, a semigroup can be embedded in a group. Since cancellation laws hold in a group and hence in any subset of it, for embeddability in a group, the semigroup needs to be cancellative as already remarked. We also remarked how the cancellation property alone is not sufficient and how Ore could impose an appropriate condition. The content of the following theorem is exactly this. The theorem is attributed to Oystein Ore in [5] and the proof follows that given by David Rees in [1].

Theorem 2.1. *Let $(S, \cdot)$ be a cancellative semigroup satisfying Ore's condition: $aS \cap bS \neq \emptyset$ for all a, b in S, then $(S, \cdot)$ is embeddable in a group.*

The Climax

Proof. Let I_S denote the set of all 1-1 partial functions on S. If a is in S, let f_a denote the function $f_a(x) = ax$ for all x in S. The function f_a is one-one since $f_a(x) = f_a(y)$ implies $ax = ay$ which by cancellation of 'a' implies $x = y$. So ${f_a}^{-1}$ is also a partial function on S. Note that I_S is a semigroup under composition of relations.

Let A denote the sub semigroup of I_S generated by i_A, all the f_a's and all the ${f_a}^{-1}$'s. Thus any element of A can be written as a finite product $f = g_{a_1}g_{a_2}...g_{a_n}$ where each g_{a_i} is f_{a_i} or ${f_{a_i}}^{-1}$ for $i = 1, 2, ..., n$. We are interested in deriving the fact that if $f = g_{a_1}g_{a_2}...g_{a_n}$ is in A then ${g_{a_n}}^{-1}{g_{a_{n-1}}}^{-1}...{g_{a_1}}^{-1}$ is also in A. While this looks like an obvious fact in view of A being generated by the f_a's and ${f_a}^{-1}$'s, there is a catch. The domain of ${g_{a_n}}^{-1}{g_{a_{n-1}}}^{-1}...{g_{a_1}}^{-1}$ could turn out to be empty in which case it would not be a partial function. Our job would be almost done if we show that $dom\ f \neq \emptyset$ for every product $f = g_{a_1}g_{a_2}...g_{a_n}$. We claim that $dom\ f$ and $ran\ f$ are right ideals of S. The proof of the claim is by induction on n. For any $a \in S$, $dom\ f_a = S$ is a right ideal of S. Moreover, $x \in ran\ f_a \Rightarrow x = f_a(s)$ for some $s \in S \Rightarrow x = as$, and hence for any $y \in S, xy = (as)y = a(sy) = f_a(sy)$. This shows that xy is in $ran\ f_a$ and hence $ran\ f_a$ is a right ideal of S. Also $dom\ ({f_a}^{-1}) = ran\ f_a$ has already been proved to be a right ideal of S. Further, $ran\ {f_a}^{-1} = dom\ f_a$ is a right ideal of S. Thus our claim is true for $n = 1$.

Let $n > 1$. Assume that the claim is true for $n - 1$. Suppose that $dom\ f \neq \emptyset$, where $f = g_{a_1}g_{a_2}...g_{a_n}$. If x is in $dom\ f = dom\ (g_{a_1}g_{a_2}...g_{a_n})$ and $y \in S$, then $x \in dom\ (g_{a_2}...g_{a_n})$ and since, by our induction assumption, $dom\ (g_{a_2}...g_{a_n})$ is a right ideal of S, $xy \in dom\ (g_{a_2}...g_{a_n})$. But $(g_{a_2}...g_{a_n})(xy) = [(g_{a_2}...g_{a_n})(x)](y)$.

Since $x \in dom\ (g_{a_1}g_{a_2}...g_{a_n})$, it is obvious that $(g_{a_2}...g_{a_n})$ $(x) \in dom\ f_{a_1}$ and because $dom\ g_{a_1}$ is a right ideal of S, it follows that $[(g_{a_2}...g_{a_n})(x)](y)$ is in $dom\ g_{a_1}$ and $dom\ (g_{a_2}...g_{a_n})(xy)$ is in $dom\ g_{a_1}$ proving that $dom\ f$ is a right ideal of S if it is nonempty.

We shall now use Ore's condition to prove that $dom\ f$ is nonempty. Let $a \in ran\ (g_{a_2}g_{a_3}...g_{a_n})$ and $dom\ g_{a_1}$. By Ore's condition there exist u, v in S such that $au = bv = w$ (say). Since $ran\ (g_{a_2}g_{a_3}...g_{a_n})$ is a right ideal

A promise redeemed.

Goal in Sight.

of S, $w = bv \in dom\ g_{a_1}$ and since $dom\ g_{a_1}$ is a right ideal of S, $w = bv \in dom\ g_{a_1}$. $w \in ran\ (g_{a_2}g_{a_3}...g_{a_n})$ implies $w = (g_{a_2}g_{a_3}...g_{a_n})(s)$ for some $s \in S$. Hence $(g_{a_2}g_{a_3}...g_{a_n})(s) \in dom\ g_{a_1}$ which means $s \in dom\ (g_{a_1}g_{a_2} ...g_{a_n})$ and hence $dom\ f \neq \emptyset$.

Now it is time to redeem an earlier promise. We shall now introduce a congruence relation θ on I_S so that partial functions that are essentially alike are treated as same. For f, g in I_S, define $f\theta g$ if and only if there is an element h in I_S such that h is a common restriction of f and g, i.e., $dom\ h \subseteq dom\ f, dom\ h \subseteq dom\ g$ and $h(x) = f(x) = g(x)$ for all x in $dom\ h$. We check the transitivity of θ and conclude that θ is an equivalence relation as θ is clearly reflexive and symmetric. Suppose $f\theta g$ and $g\theta h$. Since $f\theta g$, there exists a common restriction of f and g. There exists a common restriction v of g and h for a similar reason. Consider $w = uv^{-1}v$. We wish to show that w is a common restriction of f and h. Now, $dom\ w \subseteq dom\ v \subseteq dom\ h$. Further $x \in dom\ w \Rightarrow uv^{-1}v(x)$ is defined implies $u(x)$ is defined so that $x \in dom\ u$. Hence, $dom\ w \subseteq dom\ u \subseteq dom\ f$. Also, $x \in dom\ w \Rightarrow w(x) = uv^{-1}v(x) = u(x) = f(x)$. Also, $x \in dom\ w \Rightarrow w(x) = uv^{-1}v(x) = u(x) = g(x)$. But $g(x) = u(x)$ on $dom\ v$ and $w \subseteq dom\ v$. Therefore, $w(x) = v(x) = h(x)$. Hence, w is a common restriction of f and h and so $f\theta h$.

To show that the equivalence relation θ is a congruence relation, we have to show that $f\theta g \Rightarrow fh\theta gh$ and $hf\theta hg$ for each h in I_S. Assume $f\theta g$. Then there exists u in I_S such that u is a common restriction of f and g. But then, hu is a common restriction of hf and hg proving $hg\theta hg$ and uh is a common restriction of fh and gh proving $fh\theta gh$. This shows that θ is a congruence relation. Hence, it is meaningful to talk about the factor semigroup I_S/θ. We claim that I_S/θ is in fact a group. In I_S/ϕ, let $[f]$ denote the class containing f. If i denotes the identity function on S, $[f][i] = [fi] = [f]$. For

$[f]$ in I_S/ϕ, $[f][f^{-1}] = [ff^{-1}] = [i]$ since ff^{-1} is the restriction of i to *ran* f. Hence I_S/ϕ is a group. To show that S can be embedded in I_S/ϕ, we exhibit the mapping $a \to [f_a]$.

Ore's condition is only sufficient.

Now $f_{ab}(x) = (ab)x = a(bx) = f_a(bx) = f_a f_b(x)$ for all a, b, x in S. Suppose $[f_a] = [f_b]$. Then there exists g in I_S such that g is a common restriction of f_a and f_b. For x in *dom* f, $g(x) = f_a(x) = f_b(x)$ and so $ax = bx$. But S is cancellative. So $a = b$. Thus $a \to [f_a]$ is 1-1. □

Remark 2.2. Rees's condition is only a sufficient condition. Necessary and sufficient conditions given by Malcev in 1939 involve countably infinitely many conditions [5]. Dubreil [6] proved in 1943 that Ore's condition of the theorem is necessary as well as sufficient for embedding the semigroup in a group of right quotients of the semigroup.(A group G is called a group of right quotients of a semigroup S if G contains S and each element of G is expressible in the form st^{-1} for some elements s, t of S.)

Remark 2.3. We have already remarked on the scarcity of structure in semigroups. It is no wonder that such an amorphous entity can be embedded in a structure that involves only set theory and therefore carries minimal structure.

The theorem has been generalized to a much wider setting by Anya Katsevich and Peter Mikusinski [7]. The interested reader can refer to [7].

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Suggested Reading

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