Taylor’s theorem in analysis provides a way of approximating an \( n+1 \)-times differentiable real function by an \( n \)th degree polynomial in a neighbourhood of a point \( x_0 \). The usefulness of the theorem lies in the fact that if the bounds on \( |f^{(n+1)}(x)| \) are known, then the error introduced by the polynomial approximation can be estimated. So while Taylor’s theorem provides an extremely useful manner of approximating the given function to a desired degree of precision in a neighbourhood of a given point, it says nothing about the nature of the polynomial approximation itself, and intuitive arguments are often offered to explain why the derivatives of \( f \) must figure in the coefficients of such a polynomial. In what follows, we shall attempt a rigorous explanation of the sense in which such a polynomial is the ‘best’ approximation of the given function.

**Linear Approximations**

To begin with, let us consider a linear approximation \( g(x) = a(x - x_0) + b \) of a differentiable function \( f(x) \) at the point \( P(x_0, f(x_0)) \). Evidently if the line \( g(x) \)
While Taylor’s theorem provides an extremely useful manner of approximating the given function to a desired degree of precision in a neighbourhood of a given point, it says nothing about the nature of the polynomial approximation itself.

Instead of considering just \( \lim_{x \to x_0} g(x) \), let us consider \( \lim_{x \to x_0} \frac{g(x) - f(x)}{x - x_0} \). Clearly this limit is \( a - f'(x_0) \). If \( a \neq f'(x_0) \) then \( \lim_{x \to x_0} \frac{g(x) - f(x)}{x - x_0} \neq 0 \). Only the line passing through \( P \) with slope \( a = f'(x_0) \) would yield \( \lim_{x \to x_0} \frac{g(x) - f(x)}{x - x_0} = 0 \). Qualitatively this fact could be interpreted as follows: Whereas any line passing through \( P \) differs from \( f \), at a point \( x \) beside \( x_0 \), by a quantity which tends to zero as \( x \) tends to \( x_0 \), the line with slope \( f'(x_0) \) is the only line for which this difference is much smaller in relation to \( x - x_0 \). In this sense, \( g(x) = f'(x_0)(x - x_0) + f(x_0) \) is a better linear approximation of \( f \) at \( x_0 \) than any other linear approximation.

Let us now formalize this idea and then generalize it to polynomial approximations of any desired degree \( n \). For this we introduce the concept of infinitesimals. We shall use infinitesimals to define which of two approximations of a function is ‘better’. This will then allow us to determine the required ‘best’ approximation for the function \( f \) in a neighbourhood of a point \( x_0 \).

**Infinitesimals**

A function \( \alpha(x) \) is an infinitesimal as \( x \to a \) if \( \lim_{x \to a} \alpha(x) = 0 \) where \( a \) is either real or \( \pm \infty \).

If \( \alpha, \beta \) are infinitesimals as \( x \to a \), we say that they are of the same order if \( \lim_{x \to a} \frac{\beta}{\alpha} = A \), where \( A \) is any non-zero real number and we say that \( \beta \) is an infinitesimal of higher order than \( \alpha \) if \( \lim_{x \to a} \frac{\beta}{\alpha} = 0 \). Additionally we say that \( \beta \) is an infinitesimal of order \( k \) with respect to \( \alpha \) as \( x \to a \) if \( \lim_{x \to a} \frac{\beta}{\alpha^k} = A \neq 0 \).
Whereas any line passing through P differs from \( f \), at a point \( x \) beside \( x_0 \), by a quantity which tends to zero as \( x \) tends to \( x_0 \), the line with slope \( f'(x_0) \) is the only line for which this difference is much smaller in relation to \( x - x_0 \).

Given an arbitrary function \( f \) and two function \( g_1 \) and \( g_2 \), we say that \( g_1 \) is a better approximation of \( f \) than \( g_2 \) in a neighbourhood of \( x_0 \) if \( f - g_1 \) is an infinitesimal of higher order with respect to \( x - x_0 \) than \( f - g_2 \), as \( x \to a \).

**Theorem 1.** Let \( f \) and \( g \) be \( n \)-times differentiable functions in a neighbourhood of \( x_0 \). Then \( f - g \) is an infinitesimal of order greater than \( n \) with respect to \( x - x_0 \) as \( x \to x_0 \) if and only if \( f^{(i)}(x_0) = g^{(i)}(x_0) \) for \( i = 0, \ldots, n \).

**Proof.** L’Hôpital’s Rule states that if \( f \) and \( g \) are real and differentiable functions in \( (a, b) \) and \( g'(x) \neq 0 \) for all \( x \in (a, b) \) and \( \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \) exists then \( \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \).

Suppose \( f^{(i)}(x_0) = g^{(i)}(x_0) \) for \( i = 0, \ldots, n \). By differentiating the numerator and denominator \( n \)-times we get the following:

\[
\lim_{x \to x_0} \frac{f(x) - g(x)}{(x - x_0)^n} = \lim_{x \to x_0} \frac{f'(x) - g'(x)}{n!} = \ldots = \lim_{x \to x_0} \frac{f^{(n)}(x) - g^{(n)}(x)}{n!} = 0.
\]

Thus by applying L’Hôpital’s Rule \( n \) times we get the desired equality \( \lim_{x \to x_0} \frac{f(x) - g(x)}{(x - x_0)^n} = 0 \) and it follows that \( f - g \) is an infinitesimal of order greater than \( n \) with respect to \( x - x_0 \) as \( x \to x_0 \).

Conversely assume that \( f - g \) is an infinitesimal of order greater than \( n \) with respect to \( x - x_0 \) as \( x \to x_0 \) and that there exists a positive integer \( k \leq n \) such that \( f^{(k)}(x_0) \neq g^{(k)}(x_0) \). Let \( i \) be the smallest such positive integer.

Applying L’Hôpital’s Rule \( i \) times gives us the following:

\[
\lim_{x \to x_0} \frac{f(x) - g(x)}{(x - x_0)^i} = \lim_{x \to x_0} \frac{f'(x) - g'(x)}{i!} = \ldots = \lim_{x \to x_0} \frac{f^{(i-1)}(x) - g^{(i-1)}(x)}{i!} = \lim_{x \to x_0} \frac{f^{(i)}(x) - g^{(i)}(x)}{i!} \neq 0.
\]

It follows that \( f - g \) is an infinitesimal of order \( i \leq n \) with respect to \( x - x_0 \) as \( x \to x_0 \) contradicting the assumption that \( f - g \) is an infinitesimal of order greater than \( n \) with
respect to $x - x_0$ as $x \to x_0$.

Thus $f - g$ is an infinitesimal of order greater than $n$ with respect to $x - x_0$ as $x \to x_0$ if and only if $f^{(i)}(x_0) = g^{(i)}(x_0)$ for $i = 0, \ldots, n$.

**Best Polynomial Approximation**

We define a best polynomial approximation of degree $n$ to an $n+1$-times differentiable real function $f$ in a neighbourhood of a point $x_0$ to be the polynomial $P_n(x)$ such that $f - P_n$ is an infinitesimal of order greater than $n$ with respect to $x - x_0$ as $x \to x_0$.

Let $A_n(x)$ denote a polynomial of degree $n$. We shall now show with proof what the nature of the coefficients of $A_n(x)$ need to be for it to be the best polynomial approximation of degree $n$ of the function $f$ in a neighbourhood of the point $x_0$.

**Theorem 2.** The best polynomial approximation of degree $n$ to an $n+1$-times differentiable real function $f$ in a neighbourhood of a point $x_0$ is

$$P_n(x) = f(x_0) + f'(x_0)(x_0 - x) + \frac{f^{(2)}(x_0)}{2!}(x_0 - x)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x_0 - x)^n.$$

**Proof.** Let us consider an arbitrary $n$th degree polynomial centred at $x_0$

$$A_n(x) = a_n(x-x_0)^n + a_{n-1}(x-x_0)^{n-1} + \ldots + a_1(x-x_0) + a_0.$$

A necessary and sufficient condition for $f - A_n(x)$ to be an infinitesimal of order greater than $n$ with respect to $x - x_0$ as $x \to x_0$ is, by Theorem 1, that $A_n^{(i)}(x_0) = f^{(i)}(x_0)$ for all $i = 0, \ldots, n$.

Thus $A_n^{(i)}(x_0) = i!a_i = f^{(i)}(x_0)$ and hence $a_i = \frac{f^{(i)}(x_0)}{i!}$. A comparison with the coefficients of $P_n$ allows us to conclude that $f - P_n$ is an infinitesimal of order greater than $n$ with respect to $x - x_0$ as $x \to x_0$. If for any $i, a_i \neq \frac{f^{(i)}(x_0)}{i!}$ then clearly $f - A_n$ would be an infinitesimal of
order at most $n$ with respect to $x - x_0$. Thus if $A_n(x) \neq P_n(x)$, $A_n$ would be a worse approximation of $f$ than $P_n$.

We have thus shown that

$$P_n(x) = f(x_0) + f'(x_0)(x_0 - x) + \frac{f''(x_0)}{2!}(x_0 - x)^2 + \ldots$$

$$+ \frac{f^n(x_0)}{n!}(x_0 - x)^n$$

is the best polynomial approximation of degree $n$ of the function $f(x)$ in a neighbourhood of $x_0$, with a suitable elaboration of the sense in which such a polynomial qualifies as ‘best’ approximation.